HYDRAULICS

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HYDRAULICS

A Text on Practical Fluid Mechanics

BY

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PREFACE

The present fourth edition of this book is the result of a gradual development from the first edition which was published twenty-one years ago. The presentation has here been generalized so as to apply to all fluids, whether gases or liquids, or whether compressible or incompressible. It is evident that the fundamental principles and the general equations resulting therefrom are applicable to all fluids, and in order to adapt them to specific numerical cases all that is necessary is a knowledge of the values of certain physical properties such as viscosity and density.

The treatment here given is founded upon the principles of dimensional analysis and the basic theorems of mechanics, and much less dependence is placed upon purely empirical formulas. All of these facts entitle the book to be classified as a text on fluid mechanics. However it is fluid mechanics applied to practical engineering problems and not in the form of abstract theory the utility of which is not always apparent.

As in the case of the previous editions, the presentation has been made as brief and concise as is consistent with clearness. The principal attention has been devoted to broad fundamental principles and special cases have been introduced only where necessary and largely as illustrations of the application of the general principle. An earnest attempt has been made to divert the student from a lot of special formulas, diagrams, and charts, and instead to present to him the basic physical and mathematical considerations involved. Therefore certain convenient formulas and charts have been reluctantly omitted, despite their utility to the engineer, either because they are lacking in generality or because their use tends to prevent the student from obtaining the proper understanding of the fundamentals involved.

The author desires to express his thanks to the individuals and organizations that have furnished him with the photographs which have been used for the illustrations to which their names are appended. The author has been assisted in the preparation

of this edition by the advice and suggestions of so many users of the former editions over a period of years that it would certainly be impossible to list all of them without overlooking some and so it will not be attempted. However he is deeply grateful to all of them for their help and desires to acknowledge it here as best he can. The author does desire to express his indebtedness specifically to some of his associates at the California Institute of Technology for their advice and help. Among these are Dr. Theodor von Kármán, director of the Guggenheim Laboratory of Aeronautics, Dr. Robert T. Knapp, associate professor of hydraulic engineering, Dr. Arthur T. Ippen, instructor in hydraulics, and Mr. James W. Daily, who has assisted him in reading proof.

R. L. D.

California Institute of Technology,
Pasadena, Calif.,
August, 1937.

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NOTATION

The notation employed adheres, as far as possible, to the "Standard Symbols for Hydraulics," approved in 1929 by the American Standards Association. Since many more concepts are necessary than are provided for in the approved list, the latter had to be extended, and this brought about certain conflicts which compelled a few deviations. The symbols given in this table are generally used throughout the text. However, in some cases where it seemed desirable to use a certain notation for the treatment of a special topic, which is confined to a single article, some symbols may be used that are not in accord with the table, but such local deviations will be clearly indicated, and the usage is not to be employed elsewhere.

The equations given in this text may, for the most part, be used with any consistent system of units. The exceptions will be those equations which contain a numerical factor that is not an abstract number, in which case certain specific dimensions must be employed. In this book the engineers' foot-pound (force)-second system of units is employed except where especial convenience or common custom renders an exception necessary. However, it is not always necessary to use these particular dimensions in the general equations, and the application of the most elementary principles of dimensional analysis will enable one to determine what units are allowable in any equation.

A = any area in square feet = cross-sectional area of stream.
= area in turbines or pumps measured normal to direction of absolute velocity of fluid.

a = area in turbines or pumps measured normal to direction of relative velocity of fluid.

B = any width in feet.
= length of weir crest in feet.
= width of turbine runner or pump impeller in inches.

C = Chezy's coefficient.

C_f = friction-drag coefficient.

c = any coefficient.

c_c = coefficient of contraction.

c_d = coefficient of discharge.
c_v = coefficient of velocity.
D = diameter of pipe in inches.
= diameter of turbine runner or pump impeller in inches.
d = diameter of pipe in feet.
= depth in open channel in feet.
E = enthalpy, or heat content, in British thermal units per pound.
= linear modulus of elasticity in pounds per square inch.
E_v = volume modulus of elasticity in pounds per square inch.
e = over-all efficiency $= e_h \times e_m \times e_v$.
e_h = hydraulic efficiency.
e_m = mechanical efficiency.
e_v = volumetric efficiency.
F = force or total pressure in pounds.
f = friction factor for pipes.
G = total weight in pounds.
g = acceleration of gravity in feet per second per second = 32.174 ft./sec.^{2}
H = total effective head in feet $= z + h_p + h_v$.
h = any head in feet.
h_f = head lost in friction in feet of the fluid.
h_p = pressure head in feet of the fluid $= p/w$.
h_v = velocity head in feet $= V^2/2g$.
I = plane moment of inertia.
= internal thermal energy in B.t.u. per pound.
i = slope of bed of open channel.
J = polar moment of inertia.
= number of foot-pounds in 1 B.t.u. = 778.
K = any constant.
k = coefficient of loss.
L = length of pipe in feet.
m = hydraulic radius or, hydraulic mean depth, in feet $= A/P$.
N = revolutions per minute.
N_s = specific speed $= N\sqrt{\text{b.hp.}}/h^{5/4}$ for turbines, $= N\sqrt{\text{g.p.m.}}/h^{3/4}$ for centrifugal pumps.
n = Kutter's coefficient of roughness.
= exponent or any abstract number.
P = total pressure on area in pounds (in statics).
= power in foot-pounds per second.
= wetted perimeter in feet.
p = intensity of pressure in pounds per square foot.
Q = total volume in cubic feet.
q = rate of discharge in cubic feet per second.
R = Reynolds number $= dV\mu/\rho = dV/\nu$.
r = any radius in feet.
S = hydraulic slope $= h_f/L$.
s = specific gravity relative to some one fluid.
T = torque in foot-pounds.
t = time in seconds.

= thickness in inches.
U = uniform velocity in velocity field.
u = circumferential velocity of a point on rotating turbine runner or pump impeller in feet per second $= r\omega$.
V = mean velocity of fluid in feet per second.
= absolute velocity of fluid in turbine or pump at any point in rotating element in feet per second.
V_r = radial component of velocity in runner or impeller in feet per second $= V \sin \alpha = v \sin \beta$.
V_u = tangential component of absolute velocity in runner or impeller in feet per second $= V \cos \alpha$.
v = relative velocity in runner or impeller in feet per second.
= specific volume in cubic feet per pound.
W = weight per unit time in pounds per second $= wq$.
w = specific weight in pounds per cubic foot.
y = distance along a plane in feet.
Z = crest height of weir in feet.
z = elevation *above* any datum plane in feet (in flow).
= depth of point in liquid *below* the surface (in statics).
α = angle between V and u measured between their positive directions.
β = angle between v and u measured between their positive directions.
ϕ = ratio of u_1 to $\sqrt{2gh}$ for turbines and u_2 to $\sqrt{2gh}$ for pumps (phi).
ϕ_e = value of ϕ at point of maximum efficiency.
μ = absolute viscosity in pound-seconds per square foot (mu).
ν = kinematic viscosity in square feet per second $= \mu/\rho$ (nu).
ρ = density in slugs per cubic foot $= w/g$ (rho).
ω = angular velocity in radians per second $= 2\pi N/60$ (omega).
τ = shear per unit area in pounds per square foot (tau).

Values at specific points will be indicated by numerical subscripts. In the use of subscripts (1) and (2) the fluid is always assumed to flow from (1) to (2).

Abbreviations

g.p.m. = gallons per minute.
r.p.m. = revolutions per minute.
b.hp. = brake horsepower.
w.hp. = water horsepower.

Conversion Factors

1 U.S. gal. = 231 cu. in. = 0.13368 cu. ft. = 8.33 lb. water.
1 cu. ft. = 7.48 U.S. gal.
1 cu. ft. per sec. = 448 g.p.m.
1 lb. per sq. in. = 2.31 ft. of water at 64°F.
1 in. of mercury = 1.132 ft. of water.
1 hp. = 550 ft. lb. per sec. = 0.746 kw.

HYDRAULICS

CHAPTER I

PROPERTIES OF FLUIDS

1. Introduction.—*Fluid mechanics* is the science of the mechanics of liquids and gases and is based upon the same fundamental principles as are employed in the mechanics of solids. It deals with both compressible and incompressible fluids, or, in other words, with fluids of either variable or constant density. However, in cases where there is an appreciable change in specific volume, thermodynamics is also involved.

A fluid is considered as compressible or of variable density when, as in the case of a gas, it undergoes a decided change in specific volume as the result of a variation in pressure. While there is no such thing in reality as an incompressible fluid, this term is applied where the change in density with a variation in pressure is so small as to be negligible. This is usually the case with all liquids, even for very large pressure differences, and for gases where the variation in pressure is small relative to the total pressure of the gas.

Since all fluids are compressible in varying degrees, it is impossible to draw a definite line separating the cases where compressibility must be considered from those where it may be neglected. Water, for instance, is ordinarily treated as an incompressible fluid, yet where there are extremely large pressure differences the variation in density is appreciable. This is especially true as higher temperatures are reached. Thus water at 600°F. is about ten times as compressible as it is at 32°F. On the other hand, the flow of air in a ventilating system is an example of a case where air may be treated as incompressible, since the pressure variation is so small. But for a gas or a vapor, such as steam, flowing at high velocity in a very long pipe line the drop in pressure may be so great that compressibility cannot be ignored. For an airplane flying at speeds up to 200 m.p.h. the surrounding

air may be considered as of constant density. But as the velocity of an object through the air approaches the velocity of sound, the pressure of the air adjacent to the body becomes materially different from that of the air at some distance away, so that the air must then be treated as a compressible fluid.

To consider a fluid as compressible complicates the equations very much and renders certain solutions extremely difficult. Therefore it is customary to assume a fluid as incompressible, as long as the error in so doing is not too great. Fortunately, in many cases the error is in reality almost negligible.

Hydromechanics is literally the mechanics of water, but in practice it is not so restricted. Instead it is the mechanics of all liquids and even of gases, where compressibility effects are not large. It may be divided into three branches: *hydrostatics* is the study of the mechanics of fluids at rest; *hydrokinematics* deals with velocities independent of forces or energy; while *hydrodynamics* is concerned with the relations between velocities and the forces exerted by or upon fluids in motion.

Hydraulics is practical hydromechanics; that is, it is the study of the applications of hydromechanics to engineering problems.

By idealizing conditions it is possible to study hydrodynamics as a subject in pure mathematics, but naturally the results of such studies are often of little practical value. The determination of actual results by rigorous mathematics is often impossible because of the fact that the exact nature of certain hydraulic phenomena is either unknown or, if known, is so complex that it is not feasible to express it as a mathematical function. It is necessary, therefore, to resort to a combination of rigid mathematics, empirical expressions, and experimental coefficients. The science that results is called *hydraulics*. The modern trend is to abandon purely empirical equations that have little rational basis and are very limited in range and to develop more rational equations from fundamental theoretical considerations. This is resulting in equations which are much more general in their application and which may be applied over a very much wider range of conditions.

It is seen that hydraulics is not an exact science. In its actual applications much depends upon the judgment and the experience of the engineer. In many cases it is necessary to compute results for which satisfactory experimental data are lacking.

Also, in using any experimental factors or empirical formulas it is desirable to have some familiarity with the work upon which they are based in order to judge as to their application to the case in hand.

2. Distinction between a Solid and a Fluid.—The distinction between a solid and a fluid is ordinarily quite clear, but there are plastic solids which flow under the proper circumstances, and even metals may flow under high pressures. On the other hand, there are certain very viscous liquids which do not flow readily, and it is easy to confuse them with the plastic solids. The distinction is that any fluid, no matter how viscous, will yield in time to the slightest stress. But a solid, no matter how plastic, requires a certain magnitude of stress to be exerted before it will flow.

Also when the shape of a solid is altered by external forces the tangential stresses between adjacent particles tend to restore the body to its original figure. With a fluid these tangential stresses are proportional to the velocity of deformation and vanish as the velocity approaches zero. When motion ceases the tangential stresses disappear and the fluid does not tend to regain its original shape.

3. Distinction between a Gas and a Liquid.—A fluid may be either a gas or a liquid. A gas is quite compressible, and when all external pressure is removed it tends to expand indefinitely. A gas is, therefore, in equilibrium only when it is completely enclosed. A liquid, on the other hand, is relatively incompressible, and, if all pressure, except that of its own vapor, be removed, the cohesion between adjacent particles holds them together so that the liquid does not expand indefinitely. Therefore, a liquid may have a free surface, that is, a surface from which all pressure is removed, except that of its own vapor.

The volume of a gas is greatly affected by changes in pressure or temperature or both. It is usually necessary, therefore, to take account of changes in volume and temperature when dealing with gases. Since the mechanics of gases where volume changes are important is largely one of heat phenomena, it is usually called *thermodynamics*.

The volume of a liquid is affected to a very small extent by changes in pressure or temperature, and for most purposes the changes in volume or temperature may be ignored.

4. Perfect and Actual Fluids.—A *perfect* or *ideal* fluid is one in which the internal forces at any internal section are always normal to the section, even during motion. That is, the forces are purely pressure forces. Since there can be no tangential forces, it is, therefore, frictionless. Such a fluid does not exist in reality.[1] In an *actual* fluid, in addition to the pressure forces, tangential or shearing stresses always come into being whenever motion takes place, thus giving rise to fluid friction, since these forces oppose the sliding of one particle past another. These forces are due to a property called *viscosity*.

5. Viscosity.—The viscosity of a fluid is a measure of its resistance to shear. In Fig. 1 are shown two parallel plates a distance y apart, the space between being filled with the fluid. The lower surface is assumed to be stationary, while the upper one is moved parallel to it with a velocity V by the application of a force F corresponding to some area A of the moving plate. Particles of the fluid in contact with each plate will adhere to it, and, if the distance y is not too great or the velocity V too high, the velocity gradient will be a straight line. The action is much as if the fluid were made up of a series of thin sheets, each one of which would slip a little relative to the next one. It will then be found that

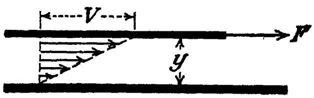

FIG. 1.

$$F = \mu A V/y,$$

where μ is the coefficient of viscosity. If τ is the shearing stress per unit area between any two thin sheets of fluid, then since the velocity gradient is a straight line,

$$\mu = \frac{F}{A}\frac{y}{V} = \tau\frac{dy}{dV}.$$

Viscosity as just defined is called *absolute* viscosity.

In the case of a solid, shear stress is proportional to the *magnitude* of the deformation; but, since for a fluid

$$\tau = \frac{F}{A} = \frac{\mu dV}{dy} = \frac{\mu V}{y},$$

[1] This is totally different from the definition of a perfect gas as given in thermodynamics. The chief characteristic of the perfect or ideal gas is that it obeys the law $pv = RT$ exactly, where v is specific volume, R is a constant, and T is absolute temperature.

it is seen that in the case of a fluid shear stress is proportional to the *time rate* of deformation.

In many problems involving viscosity it is found that density also enters, and therefore it is often convenient to use a value that is a combination of the two. Absolute viscosity divided by density is called *kinematic* viscosity. This designation is applicable because force is not considered, the only quantities involved being distance and time, as in kinematics.

If L is the unit of length, T the unit of time, and F the unit of force, then in the gravitational or engineers' system of units the dimensions involved in viscosity are seen to be $FL^{-2}T$. But if M is the unit of mass, then in the absolute or physicists' system of units the dimensions of viscosity are $ML^{-1}T^{-1}$, since $F = MLT^{-2}$.

Since density is $FL^{-4}T^2$ in the engineers' system and ML^{-3} in the absolute system, it may be seen that the dimensions of kinematic viscosity are L^2T^{-1} in either system of units.[1]

TABLE I.—VISCOSITY UNITS

Symbol	Quantity	Absolute system	Engineers' system
M	Mass	M	$FL^{-1}T^2$
F	Force	MLT^{-2}	F
ρ	Density	ML^{-3}	$FL^{-4}T^2$
w	Specific weight	$ML^{-2}T^{-2}$	FL^{-3}
μ	Absolute viscosity	$ML^{-1}T^{-1}$	$FL^{-2}T$
ν	Kinematic viscosity	L^2T^{-1}	L^2T^{-1}

The unit of viscosity is obtained by making $\mu = 1$ in the expression $\mu = FL^{-2}T = ML^{-1}T^{-1}$. Since four systems of units are commonly recognized, there will be that many different values of viscosity units. In the engineers' system the unit of viscosity will be 1 *lb. sec. per sq. ft.* (= 1 slug per ft. sec.), or 1 *gram sec. per sq. cm.* (= 1 metric slug per cm. sec.). In the absolute system, the unit of viscosity will be 1 *poundal sec. per sq. ft.*

[1] Density is mass per unit volume, while specific weight is weight per unit volume. In the engineers' system, density is in slugs per cubic foot in English units, for instance, while specific weight is in pounds per cubic foot, and force is in pounds. In the absolute system, density is in pounds per cubic foot, and force in poundals, or in grams per cubic centimeter and dynes, respectively. In dealing with viscosity it is essential to be consistent in the system of units used.

(= 1 lb. per ft.-sec.), or 1 *dyne sec. per sq. cm.* (= 1 gram per cm.-sec.). One dyne sec. per sq. cm. is called a *poise*, and 0.01 poise is called a *centipoise*. There are no names for the other three units.

The unit of kinematic viscosity is obtained by dividing unit absolute viscosity by unit density. It is 1 *sq. ft. per sec.*, or 1 *sq. cm. per sec.* The unit has no name in the English system or any commonly recognized one in the metric system, but the term *stoke* is sometimes employed. One stoke equals 1 poise divided by 1 gram per cu. cm., and 1 *centistoke* is 1 centipoise divided by 1 gram per cu. cm.

In computing kinematic viscosity it is necessary to be consistent. Thus in the English system, viscosity in pound seconds per square foot should be divided by density in slugs per cubic foot, or viscosity in poundal seconds per square foot should be divided by the density in pounds per cubic foot.

Some useful equivalents are:

1 poise = 0.0672 poundal sec. per sq. ft.
= $0.0672/g$ (= 0.00209) lb. sec. per sq. ft.
1 poundal sec. per sq. ft. = 14.88 poises
1 lb. sec. per sq. ft. = $14.88 \times g$ (= 478.8) poises
1 sq. cm. per sec. = 0.001076 sq. ft. per sec.
1 sq. ft. per sec. = 929 sq. cm. per sec.

Since viscosity is usually tabulated in centipoises, while the engineer usually has specific weights given in pounds per cubic foot, the following relationship will be useful:

$$\text{Kinematic viscosity in square foot per second} = \frac{0.000672 \times \text{centipoises}}{\text{lb. per cu. ft.}}$$

Absolute viscosities of various fluids are shown in Fig. 2, and their kinematic viscosities are shown in Fig. 142 (page 209). Since the viscosity of water at 68.4°F. (20.2°C.) is 1 centipoise, the centipoise scale is also one of viscosity relative to water at that temperature. Therefore the value in centipoises is numerically equal to *specific viscosity* relative to that of water at 68.4°F.

The absolute viscosity of either a liquid or a gas is practically independent of pressure for the range that is ordinarily encountered in engineering work. But for extremely high pressures the

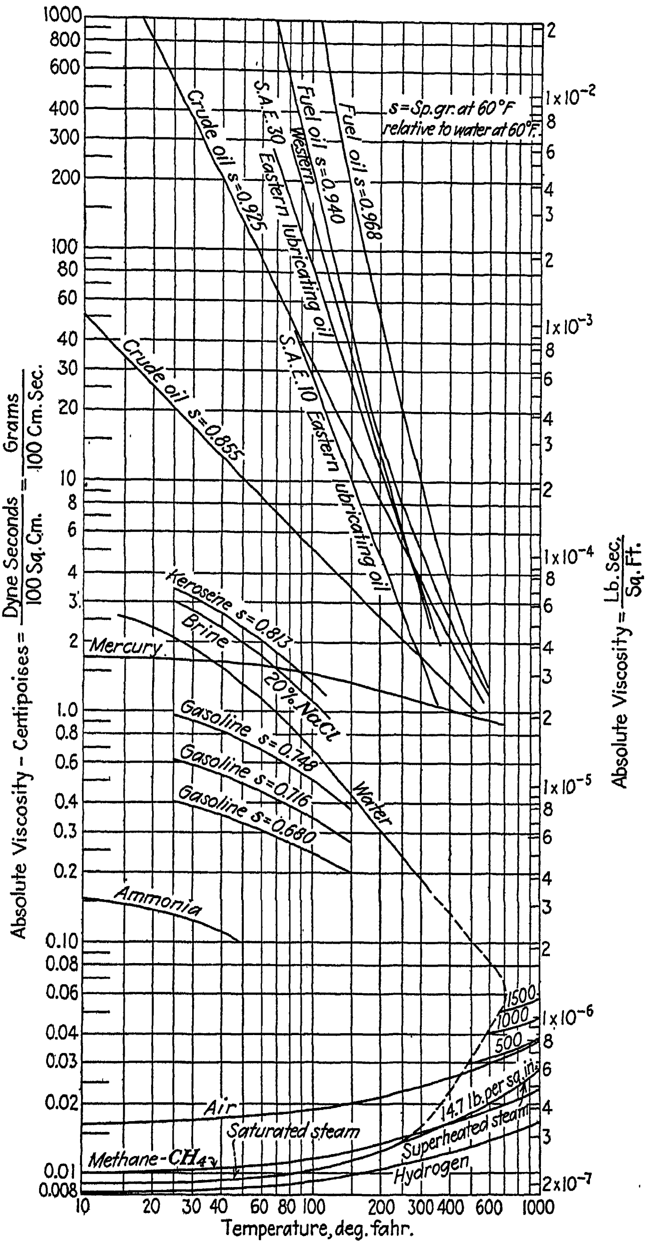

FIG. 2.—Values of absolute viscosity.

values might be materially higher than shown here. It may be seen that, as the temperature increases, the viscosities of all liquids decrease while the viscosities of all gases increase. This is because the force of cohesion, which diminishes with temperature, predominates with liquids; while with gases the predominating factor is the interchange of molecules between the layers of different velocities, and molecular activity increases with temperature.

The direct measurement of viscosity is difficult, and, therefore, for ordinary purposes certain instruments are used which measure directly some other quantity which in turn is a function of viscosity. For instance, the usual instrument used in this country for oils is the Saybolt Universal viscosimeter with which is determined the number of seconds required for 60 cc. to flow out through a small short tube. For very viscous oils the Saybolt Furol viscosimeter is used. It has a larger tube so that the time is only about one-tenth as long as that required for the Universal. Kinematic viscosity as determined by these instruments is expressed as Saybolt seconds Universal (or Furol), but an empirical equation determined by calibration must be employed to convert this value in seconds into true kinematic viscosity. There are other instruments used for oils also, and still others used for other fluids. Some of these instruments measure quantities that are proportional to absolute viscosity and others measure quantities that are proportional to kinematic viscosity.

6. Some Properties of Liquids.—Liquids have properties such as cohesion and adhesion, both of which are forms of molecular attraction. *Cohesion* enables a liquid to resist a slight tensile stress, while *adhesion* enables it to adhere to another body. *Surface tension* is due to cohesion between particles at the surface. *Capillarity* is due to both cohesion and adhesion. When the former is of less effect than the latter, the liquid will "wet" a solid surface with which it is in contact and rise at the point of contact; if cohesion predominates, the liquid surface will be depressed at the point of contact. For example, capillarity causes water to rise in a glass tube, while mercury is depressed below the true level, as is shown in the insert in Fig. 3, which is drawn to scale and reproduced full size.

The capillary rise (or depression) is determined by the equation $2T = (w_1 - w_2)bh$, where T is surface tension in units of force

per unit length, w_1 is specific weight of fluid below the meniscus, w_2 is specific weight of fluid above the meniscus, b is radius of curvature at the bottom (or top) of the meniscus, and h is capillary rise (or depression). The practical difficulty in applying this equation is that it is impossible to measure the radius of curvature of the liquid surface, and therefore certain assumptions and approximations are necessary. Thus, if the diameter of the tube is less than 0.1 in., the meniscus may be assumed to be spherical with a radius $b = \frac{r}{\cos\theta}$, where r is radius of tube and θ is contact angle between liquid and tube. If the surface is

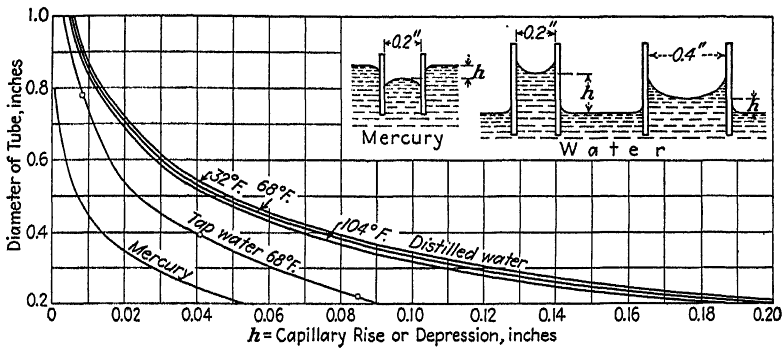

FIG. 3.—Capillarity in circular glass tubes.

clean, this angle is 0 deg. for water and about 140 deg. for mercury. For larger diameters no such simple solution is possible.[1]

Surface tension decreases with increasing temperature, but Fig. 3 shows that for water the effect of change in temperature upon capillary height is rather small. Pressures are often measured by the height to which some liquid will rise in a tube, and Fig. 3 shows the amount of error involved in using small tubes. Although the capillary effect is less for mercury than for water, if a mercury reading is multiplied by the specific gravity of mercury, the error will be greater than for a water column in the same size

[1] The curve for tap water in Fig. 3 was determined experimentally at California Institute of Technology by R. G. Folsom. He found dirty water to give even slightly lower values. The curves for pure water and for mercury were computed by the author by methods suggested by Adam in "The Physics and Chemistry of Surfaces" based upon the analysis of Lord Rayleigh in his *Scientific Papers*, vol. VI, p. 350.

tube. The curves in Fig. 3 are for water or mercury in contact with air, in which case w_2 is specific weight of air. If mercury is in contact with water, the surface tension is then less than when it is in contact with air, but in this case w_2 is specific weight of water. It so happens that this change in $(w_1 - w_2)$ is practically in the same proportion as the change in the surface tension, so that the values of the capillary depression for mercury and air are almost identical with those for mercury and water.

7. Compressibility of Water.—Water is usually said to be incompressible, and it is relatively so when compared with gases. For instance, a gas at atmospheric pressure is 20,000 times as compressible as water, while at 300 lb. per sq. in. it is still 1,000 times as compressible. But on the other hand, water is about 100 times as compressible as mild steel.

TABLE II.—VALUES OF E_v, LB. PER SQ. IN.*

Pressure, lb. per sq. in.	Temperatures, °F.				
	32°	68°	120°	200°	300°
15	292,000	320,000	332,000	308,000	
1,500	300,000	330,000	342,000	319,000	248,000
4,500	317,000	348,000	362,000	338,000	271,000
15,000	380,000	410,000	426,000	405,000	350,000

* DAUGHERTY, R. L., "Some Physical Properties of Water and Other Fluids," *Trans. A.S.M.E.*, vol. 57, no. 5, July, 1935.

The compressibilities of various substances are inversely proportional to their volume moduli of elasticity, where volume modulus of elasticity is $E_v = -V(dp/dV)$, in which V is volume and p is unit pressure. The change in volume may be either isothermal or adiabatic, thus giving isothermal or adiabatic values of E_v. For water at ordinary temperatures the difference between these two is slight. In Table II are shown a few values of the isothermal modulus for water at various temperatures and pressures. It is found that at any temperature the value of E_v increases continuously even up to 12,000 atmospheres but that at any one pressure the value of E_v is a maximum at about 120°F. In other words, water has a minimum compressibility at about 120°F. Since V/dV is an abstract ratio, the units for E_v must correspond to those used for p.

As an approximation one may use

$$V_1 - V_2 = \frac{V_1(p_2 - p_1)}{E_{vm}},$$

where E_{vm} = a mean value for the pressure range involved. Assuming it to have an average value of 300,000 lb. per sq. in., it may be seen that increasing the pressure on water by 1,000 lb. per sq. in. will compress it only about $\frac{1}{300}$, or one-third of 1 per cent of its initial volume. Therefore, it is seen that, unless the

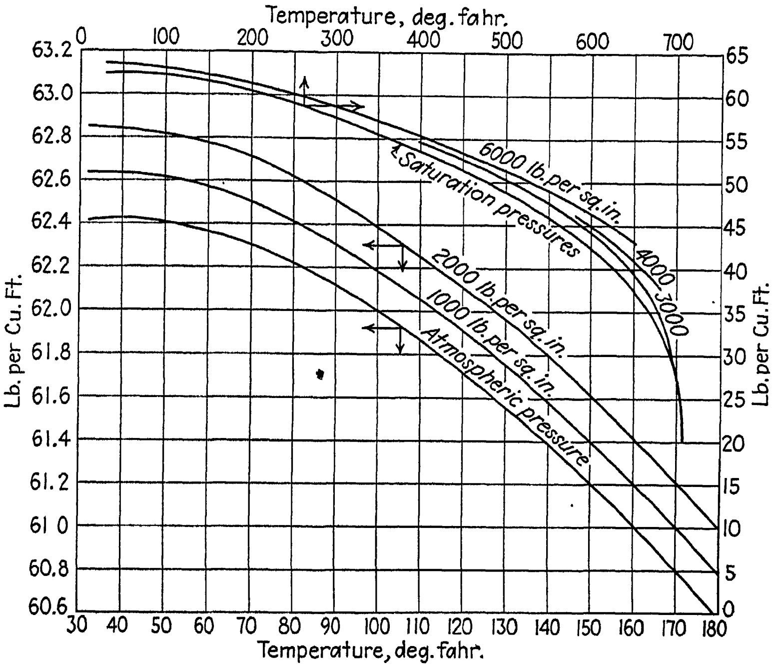

FIG. 4.—Specific weight of pure water.

pressure range is very great, the usual assumption regarding water as incompressible is justified from the practical standpoint.

8. Weight of Water.—Although the density or specific weight of water changes only slightly with pressure, its variation with temperature is quite appreciable, as may be seen in Fig. 4. The presence of dissolved air, salts in solution, and suspended matter will increase these values a very slight amount. Ocean water may ordinarily be assumed to weigh 64.0 lb. per cu. ft. Unless otherwise specified or implied, the value to use for water in the

problems in this text is $w = 62.4$ lb. per cu. ft. Of course, if another fluid is involved, the appropriate value for w must be employed.

TABLE III.—SOME PROPERTIES OF WATER AND MERCURY

Temp., °F.	Water: Specific weight w = lb. per cu. ft. at 14.7 lb. per sq. in.	Water: Specific weight w = lb. per cu. ft. at 1,000 lb. per sq. in.	Water: Absolute viscosity, poises	Water: Kinematic viscosity, sq. ft. per sec.	Mercury: Specific weight w = grams per cu. cm.	Mercury: Specific weight w = lb. per cu. ft.
32	62.42	62.64	0.01794	1.93×10^{-5}	13.595	848.7
39.2	62.43	62.64	0.01568	1.69	13.586	848.1
50	62.41	62.62	0.01310	1.41	13.571	847.2
60	62.37	62.57	0.01130	1.22	13.557	846.3
68.4	62.31	62.51	0.01000	1.09	13.546	845.7
80	62.22	62.41	0.008616	0.932	13.530	844.7
90	62.12	62.31	0.007635	0.825	13.516	843.8
100	62.00	62.19	0.006821	0.739	13.503	843.0
140	61.39	61.57	0.004699	0.515	13.449	839.6
212	59.83	60.01	0.002839	0.319	13.352	833.6
300	.....	57.54	0.00186	0.218	12.881	804.2

9. PROBLEMS

1. What is the ratio of the viscosity of water at a temperature of 70°F. to that of water at 200°F.? What is the ratio of the viscosity of the crude oil shown in Fig. 2, the specific gravity of the oil being 0.925, to that of the gasoline, specific gravity 0.680, both being at 60°F.? In cooling from 250 to 80°F., what is the ratio of the change of viscosity of the SAE 30 western oil to that of the SAE 30 eastern oil? *Ans.* 3.33, 300, 1.84.

2. At 60°F. what is the kinematic viscosity of the gasoline in Fig. 2, the specific gravity of the gasoline being 0.680? Give answer in both metric and British units.

Ans. 0.00442 sq. cm. per sec., 0.00000475 sq. ft. per sec.

3. To what temperature must the heaviest fuel oil shown in Fig. 2 be heated in order that its viscosity may be three times that of water at 40°F.?

Ans. 375°F.

4. The empirical equation for the Saybolt Universal viscosimeter is *kinematic viscosity in sq. cm. per sec.* $= 0.00220t - 1.80/t$, where t = number of seconds. The specific gravity of an oil is usually specified at 60°F. relative to water at 60°F. but may be converted approximately to any

other temperature by assuming a coefficient of expansion of 0.0004 per degree Fahrenheit. If the Saybolt viscosity of an oil whose specific gravity is 0.90 at 60°/60° is 100 Saybolt seconds at a temperature of 140°F., what is its kinematic viscosity and what is its absolute viscosity in both systems of units? *Ans.* 0.202 sq. cm. per sec. or 0.000217 sq. ft. per sec., 0.175 poise, 0.000366 lb. sec. per sq. ft.

5. The static pressure is such that water should rise in a glass tube to a height of exactly 3 in. If the diameter of the tube is 0.2 in. and the temperature 68°F., what will be the height of the water if the latter is pure? If the diameter of the tube were 0.6 in., what would be the height?

Ans. 3.21 in., 3.03 in.

6. Water at 68°F. stands in a glass tube of 0.02 in. diameter at a height of 4.50 in. Surface tension is 72.75 dynes per cm., or 0.000417 lb. per in. What is the true static height? The height of a mercury column at 68°F. is observed in a glass tube of 0.3 in. in diameter. If this reading is to be converted into height of water by multiplying by the ratio of the densities of the two liquids, what is the error? *Ans.* 2.2 in., −0.38 in. of water.

7. A cubic foot of water initially under a pressure of 20 lb. per sq. in. is subjected to a pressure of 9,020 lb. per sq. in. What is its approximate final volume? *Ans.* 0.97 cu. ft.

8. What will be the specific weight of water in the ocean at a depth of 5 miles below the surface where the pressure will be 11,930 lb. per sq. in., assuming the surface value to be 64.0 lb. per cu. ft. and that the volume modulus has an average value of 340,000 lb. per sq. in. for that pressure range? *Ans.* 66.24 lb. per cu. ft.

9. A vessel contains 3 cu. ft. of water at 50°F. and atmospheric pressure. If it is heated up to 160°F., what will be the percent change in its volume? What weight must be removed to maintain the volume at its original value?

Ans. 2.3 per cent, 4.2 lb.

CHAPTER II

INTENSITY OF PRESSURE

10. Definition of Intensity of Pressure.—By intensity of pressure is meant pressure per unit area. It may be expressed in various units such as pounds per square inch, pounds per square foot, or, as will be seen later, in feet of water, inches of mercury, etc.

If P represents the total pressure on some finite area A, while dP represents the total pressure on an infinitesimal portion of area dA, the intensity of pressure is

$$p = \frac{dP}{dA}. \tag{1}$$

If the pressure is uniformly distributed over the area in question, the intensity of pressure would then be $p = P/A$. If the pressure is not uniformly distributed, the latter expression will give the average value only.

The word "pressure" is usually used for "intensity of pressure," though the latter term should be employed where there is any possibility of misunderstanding. The word "pressure" is also used to designate the resultant force exerted on an area. In order to distinguish clearly this usage from intensity, it would be well to employ the term "resultant pressure" or "total pressure."

11. Variation of Pressure in a Liquid.—Consider a slender prism of the liquid in Fig. 5 as a free body in equilibrium. The forces acting upon it will be the pressures on its various faces and the pull of gravity. If the intensity of pressure at M be denoted by p_1, the total pressure on the end at M will be $p_1\,dA$, where dA is the cross-sectional area. In similar manner the total pressure on the end of N will be $p_2\,dA$. The weight of the volume of liquid is evidently $w\,dA\,l$. Since the prism of water is in equilibrium, the algebraic sum of the components in any direction of all the forces acting on it will be zero. If the forces be resolved along the axis MN, the three forces mentioned will be the only ones

that will appear, since the forces acting on the sides of the prism are all normal to the axis. Hence,

$$p_1\,dA - p_2\,dA + w\,dA\,l\cos\alpha = 0.$$

Since $l\cos\alpha = z_2 - z_1$, it follows that

$$p_2 - p_1 = w(z_2 - z_1). \tag{2}$$

This equation shows that the difference in the intensity of pressure at two different points varies directly as the difference in the depths of the two points. In other words, the pressure varies directly as the depth. If z is measured from the level where the pressure is zero, then

$$p = wz. \tag{3}$$

The foregoing, however, are strictly true only for an incompressible fluid in which the specific weight w is constant. For most purposes all liquids may be so considered, except when the change in pressure is enormously great. But the preceding equations should not be used for gases, except where there are relatively small differences in pressure.

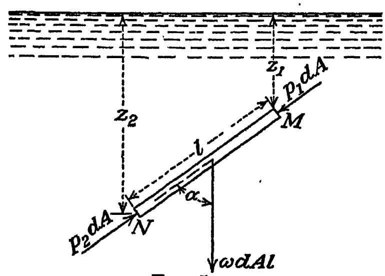

Fig. 5.

In general, Eq. (2) may be written as $dp = w\,dz$; and if w is not constant, the equation can be integrated only if the law of variation is known. As a very special case consider a perfect gas at a uniform temperature. Then $p/w = \text{const.} = p_0/w_0$, where the latter are any two known simultaneous values. Therefore, $dz = dp/w = (p_0/w_0)dp/p$, which integrates as

$$z_2 - z_1 = \left(\frac{p_0}{w_0}\right)\log_e\left(\frac{p_2}{p_1}\right).$$

12. Surface of Equal Pressure.—It may be seen from Eq. (3) that all points in a connected body of fluid at rest are under the same intensity of pressure if they are at the same depth. This indicates that a surface of equal pressure is a horizontal plane. Strictly speaking, it is a surface everywhere normal to the direction of gravity, and it is, therefore, approximately a spherical surface concentric with the earth. But a limited portion of such a surface is practically a plane area.

13. Pressure the Same in All Directions.—In a solid, owing to the existence of tangential stresses between adjacent particles, the stresses at a given point may be different in different directions. But in a fluid at rest no tangential stresses can exist, and the only forces exerted between adjacent surfaces are pressure forces normal to the surfaces. Therefore, the intensity of pressure at a given point is the same in every direction.

This can be proved by reference to Fig. 6, where there is a small triangular element of volume whose thickness perpendicular to the plane of the paper is constant and equal to dz. Let α be any angle, p the intensity of pressure in any direction, and

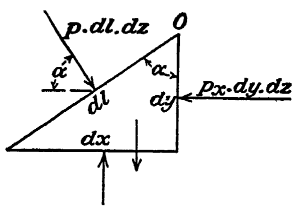

Fig. 6.

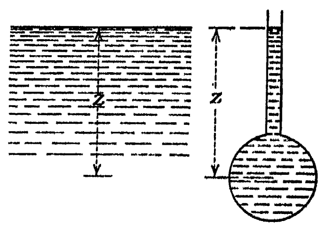

Fig. 7.

p_x the intensity of pressure on a vertical plane. The following forces act upon this volume of fluid: The pressure on the vertical face is $p_x\ dy\ dz$, the pressure on the slanting face is $p\ dl\ dz$; then there are the pressures on the horizontal face and on the two faces parallel to the plane of the paper, and the weight of the volume. Their values are not required. Since this volume is a fluid body at rest, there are no other forces besides these normal to the surfaces, and, since it is a condition of equilibrium, the sum of the components in any direction is equal to zero. Writing such an equation for components in a horizontal direction, we have only:

$$p\ dl\ dz \cos \alpha - p_x\ dy\ dz = 0.$$

Since $dy = dl \cos \alpha$, it is seen that

$$p = p_x.$$

This result is independent of the angle α, and, therefore, it follows that the intensity of pressure is the same upon any plane passing through O.

14. Pressure Expressed in Height of Liquid.—In Fig. 7 imagine a body of liquid upon whose surface there is no pressure, though

in reality the minimum pressure possible upon any liquid surface is the pressure of its own vapor, which is a function of the temperature. Then by Eq. (3) the intensity of pressure at any depth z is $p = wz$. For a given liquid w is constant, and thus there is a definite relation between p and z. That is, any pressure per unit area is equivalent to a corresponding height of liquid. In hydraulic work it is often more convenient to express intensity of pressure in terms of height of a column of fluid rather than in pressure per unit area.

Even if the surface of the liquid in Fig. 7 is under some pressure, the relation stated is still true; for this pressure on the surface could be expressed in terms of height of the liquid, and such value added to z. The resulting value of p would thus be increased by the amount of this surface pressure.

The intensity of pressure expressed in terms of the height of a column of liquid will be

$$z = \frac{p}{w}. \tag{4}$$

This relationship is true for any consistent system of units. If p is in pounds per square foot, w must be in pounds per cubic foot, and then z will be in feet of the fluid. Since intensity of pressure is commonly expressed in pounds per square inch, and since the value of w for water is commonly assumed to be 62.4 lb. per cu. ft. (see Fig. 4), there is the special relationship

$$z = 144 \times (\text{lb. per sq. in.})/62.4 = 2.308 \times (\text{lb. per sq. in.})$$

$$(\text{lb. per sq. in.}) = \frac{62.4z}{144} = 0.4333z,$$

but it must be remembered that this is true for that one specific weight only.

EXAMPLES

10. Neglecting the pressure of the atmosphere upon the surface and the compressibility of the water, what is the pressure in pounds per square inch at a depth of 2,000 ft. of water at a temperature of 39.3°F.? What is the pressure due to 54 in. of mercury at 100°F.?

Ans. 867 lb. per sq. in., 26.4 lb. per sq. in.

11. A pump for a hydraulic press delivers water at 100°F. at a pressure of 6,000 lb. per sq. in. To what height is that equivalent, assuming the specific weight were to increase uniformly with the depth?

Ans. 13,850 ft.

12. A pump in a steam plant is required to pump water at 500°F. and a pressure of 3,000 lb. per sq. in. To what height of water at constant density for these conditions is that equivalent? Another pump delivers water at 160°F. against a head of 20 ft. What is the pressure in pounds per square inch? *Ans.* 8,640 ft., 8.47 lb. per sq. in.

13. A column of mercury at 70°F. and 42 in. high is equivalent to what height of water at 200°F. (Table III)? *Ans.* 49 ft.

14. What will be the pressure at a depth of 5 miles in the ocean, neglecting increase in specific weight with depth? Assuming that the increase in specific weight is directly proportional to the depth and that it is 66.24 lb. per cu. ft. at a depth of 5 miles, what will be the pressure?

Ans. 11,750 lb. per sq. in., 11,950 lb. per sq. in.

15. Vacuum.—Pressures less than that of the atmosphere are usually called *vacuums*, a perfect vacuum meaning an entire absence of all pressure. Vacuum is usually measured from the pressure of the atmosphere as a base and is commonly, though not necessarily, measured in inches of mercury. If the atmospheric pressure is 30 in. of mercury, a perfect vacuum would then be a vacuum of 30 in. And a vacuum of 10 in. of mercury would mean that there was a real pressure of 20 in. of mercury.

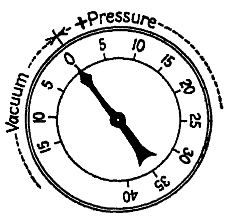

FIG. 8.—Compound gage.

EXAMPLE

15. The barometer reads 30 in. of mercury, and within a certain vessel there is a vacuum of 22 in. of mercury. What is the real pressure within that vessel in pounds per square inch? What is the excess external pressure on the walls of the vessel in pounds per square inch? *Ans.* 3.92, 10.78.

16. Absolute and Relative or Gage Pressures.—If the pressure is measured above absolute zero pressure, it is called absolute pressure. If it is measured from the atmospheric pressure as a base, it is called relative or gage pressure, since it is only relative pressure that a gage measures. Thus Fig. 8 shows a compound gage which will measure pressures either above or below that of the atmosphere. When the gage is open to the atmosphere, the hand points to zero. If the gage is connected to any vessel in which there is a pressure above that of the surrounding air, the hand will turn in a clockwise direction from zero. If the pressure is a vacuum, the hand will move in the opposite direction. Thus the gage measures only the difference between the pressure on the inside of the gage tube and that of the air surrounding the gage.

In Fig. 9 let O indicate entire absence of all pressure or absolute zero, and the ordinate OA represent the pressure of the atmosphere. Then suppose that there is any pressure such as at B. The gage will read the value AB, and this is the gage pressure. The absolute pressure is OB. Also, if there is a vacuum of AC, the gage pressure is $-AC$, the minus sign merely indicating a value below atmospheric just as a plus sign indicates a pressure above that of the atmosphere. But the absolute pressure is OC.

When dealing with absolute pressure, all values are positive. In the case of gage pressures only values above that of the atmospheric pressure are positive, but the minus sign for pressures below that of the atmosphere serves only to indicate a vacuum. There may still be a real pressure between adjacent particles of water. A true negative pressure would mean that the water was in a state of tension; and as water can sustain only a very slight tensile stress, it is impossible to have a pressure below absolute zero. Absolute zero is the point where the stress in the liquid would change from compression to tension.

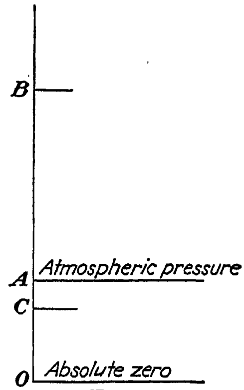

FIG. 9.

In most problems in hydraulics absolute pressures are not of special interest. The important factor is usually the difference between the pressure inside a vessel and that outside, for example, and in general this would be the gage pressure. And in many other cases the atmospheric pressure acts alike at all points and balances out. Hence gage pressures only are usually used.

EXAMPLE

16. A gage reads 50 lb. per sq. in. when the gage itself is surrounded by the atmosphere. If the air surrounding the gage be exhausted to a vacuum of 25 in. of mercury while the real pressure of the fluid on the inside of the gage tube remains the same, what will be the reading of the gage?

Ans. 62.25 lb. per sq. in.

17. Barometer.—The absolute pressure of the atmosphere is measured by the barometer. If a tube, such as that in Fig. 10, has its lower end immersed in a liquid which is exposed to the atmospheric pressure, and if air is exhausted from the tube,

the liquid will rise in it. If the air were completely exhausted, the only pressure on the surface of the liquid in the tube would then be that of its own vapor pressure, and the liquid will have reached its maximum height.

By Art. 12 it may be seen that the intensities of pressure at o within the tube and at a at the surface of the liquid outside are the same. That is, $p_o = p_a$. If the vapor pressure on the surface of the liquid in the tube were negligible, then by Eq. (3) the intensity of pressure at o is

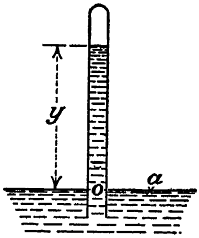

FIG. 10.—Barometer.

$$p_o = wy = p_a.$$

The liquid employed is usually mercury, because its density is sufficiently great to enable a reasonably short tube to be used, and also because its vapor pressure is negligibly small at ordinary temperatures. If water were used, the tube necessarily would be so high as to be inconvenient; and since its vapor pressure at ordinary temperatures is appreciable, a nearly perfect vacuum at the top of the column would not be attainable. The height attained by the water would consequently be less than the true barometric height and would necessitate a correction being applied to the reading. In order to minimize the error due to capillarity, the diameter of the tube should be at least 0.5 in., as may be seen by Fig. 3.

The pressure of the atmosphere is different in different localities, depending upon the elevation, and also at a given point it varies from time to time according to weather conditions. In round numbers the pressure of the atmosphere may be taken as 14.7 lb. per sq. in., 30 in. of mercury, and 34 ft. of water, but these values are not exact equivalents.

EXAMPLES

17. If the pressure of water vapor at 80°F. is 0.505 lb. per sq. in. abs., what would be the height of the water barometer, if the atmospheric pressure were 14.7 lb. per sq. in.? (Use correct specific weight of water for this temperature.) *Ans.* 32.9 ft.

18. (*a*) Assuming the specific weight of air to be 0.0807 lb. per cu. ft., what would be the height of the air surrounding the earth and producing the pressure of 14.7 lb. per sq. in. at its surface, if air were incompressible? (*b*) Assuming that the air is of uniform temperature but that its specific weight is not constant, at what heights above the earth will the pressure be

10, 5, 3, 2, 1 lb. per sq. in. (Note that in this case z increases as p decreases.)

Ans. (*a*) 26,250 ft., (*b*) 10,200; 28,400; 41,800; 52,500; 70,700 ft.

19. If the barometer reads 30 in. of mercury and a vacuum gage reads 5 in. of mercury, what is the absolute pressure?

18. Instruments for Measuring Pressure. *Gage.*—The familiar pressure or vacuum gages have already been referred to, and the combination of the two is shown in Fig. 8. In this type of instrument a curved tube is caused to change its curvature by changes of pressure within the interior of the tube. The moving end of the tube then rotates a hand by means of some intermediate links. It is usually assumed that the pressure indicated by the gage is that existing at the center of the gage. Thus the location of the center of the gage should always be taken into consideration. For instance, referring to Fig. 11, the pressure at A is the gage reading plus the distance z. If the gage reads in pounds per square inch, as is customary,

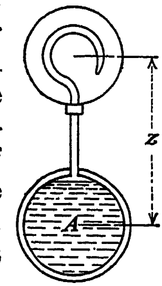

FIG. 11.—Bourdon gage.

$$\frac{p_A}{w} = \frac{144}{w} \times (\text{lb. per sq. in.}) + z,$$

where p_A/w is the height in feet of a fluid of specific weight w which would produce the same pressure as in A.

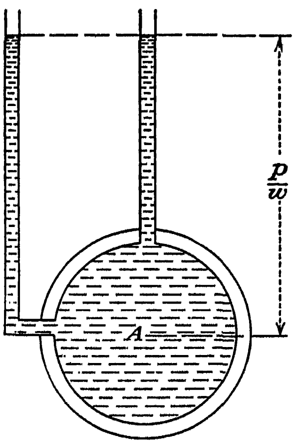

FIG. 12.—Piezometer.

Piezometer Tube.—A piezometer (which means pressure-measuring) tube is shown in Fig. 12 and is a very simple device for measuring moderate pressures. It consists of a tube in which the liquid can freely rise, without overflowing, until equilibrium is established. To reduce capillary error the diameter should preferably be at least 0.5 in. The height of the liquid in the tube will give the pressure directly. In case the fluid whose pressure is being measured is not at rest, special precautions must be taken in making the connections for the piezometer tube or any other pressure-measuring device. A discussion of this phase of the problem will be found in Art. 86.

Mercury U-tube.—For high pressure the piezometer is not suited, and some modification must be adopted. The mercury U-tube shown in Fig. 13 may then be used. If w' is the specific weight of the mercury (or whatever liquid may be employed), the pressure at the point C is $w'y$. This is also the pressure at B; but the pressure at A is greater than this by the amount $w''z$, where w'' is the specific weight of the fluid in the connecting tube $A'B$ which is assumed to be completely filled by it. Thus the pressure at A is

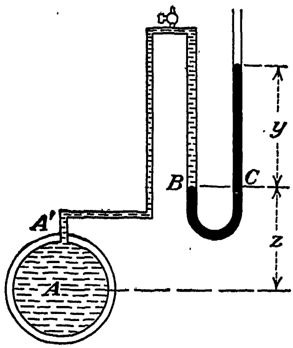

Fig. 13.

$$p_A = w''z + w'y.$$

If the vessel A contains a fluid of specific weight w, and it is desired to express the pressure in terms of the height of this fluid, then

$$\frac{p_A}{w} = \frac{w''}{w}z + \frac{w'}{w}y = s''z + sy,$$

where s and s'' are the specific gravities of the measuring liquid and the fluid in the connecting pipe, respectively, *relative to the fluid* in A. In practically all cases the connecting tube will contain the same fluid as is in A, in which case the only difference in specific weights would be due to a difference in temperature. If this is negligible, as is often the case, then $s'' = 1$, and

$$\frac{p_A}{w} = z + sy.$$

It is customary for specific gravity to denote the ratio of the specific weight of the fluid to that of water in the case of liquids and to that of air in the case of gases, but it is not so restricted in its use here.

If vessel A contained water, for example, and the connecting tube $A'B$ were filled with air, the term $w''z$ would be negligible compared with other quantities because of the relatively small value of w''. But it would be difficult to insure that the tube would be completely filled with air and, if it were partially filled with air and partially with water, an accurate computation would be impossible, unless it were known just where the water surface

in the tube was. It is therefore desirable to provide some means by which all the air may be permitted to escape and its place taken by the water or whatever other fluid might be involved.

In measuring a vacuum for which the arrangement in Fig. 14 might be used, the reading y must be interpreted as a negative quantity so that, if the connecting tube is filled with the fluid, just as in the preceding case, the result is

$$p_A = w''z - w'y$$

or

$$\frac{p_A}{w} = s''z - sy.$$

And again, if the temperature difference is not too great, this reduces to

$$\frac{p_A}{w} = z - sy.$$

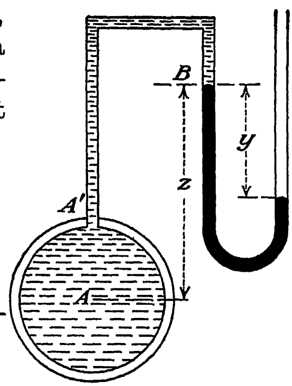

FIG. 14.

If the connecting tube were completely filled with air while water or some other dense fluid were being measured, the correction for the height z might be negligible. But it is difficult to insure this tube's being filled with air, though it is not so hard to bring it about as in the case of a pressure manometer. The arrangement in Fig. 15 is much better, as that permits no air to collect in the tube and introduce an error. In this case z is also negative, so that

$$\frac{p_A}{w} = -(s''z + sy).$$

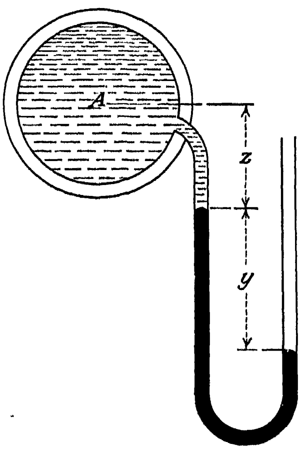

FIG. 15.

Of course, other liquids might be used in all of these instruments instead of mercury. As the specific gravity of the measuring liquid becomes less, the reading becomes larger for a given pressure, thus increasing the accuracy of the instrument.

Differential Gage.—The differential gage is used for measuring differences of pressure only. One form of this is shown in

Fig. 16(*a*). Assuming the entire connecting tubing to be filled with a fluid of specific weight w'', except that portion of the U that is filled with the denser liquid, such as mercury, for instance,

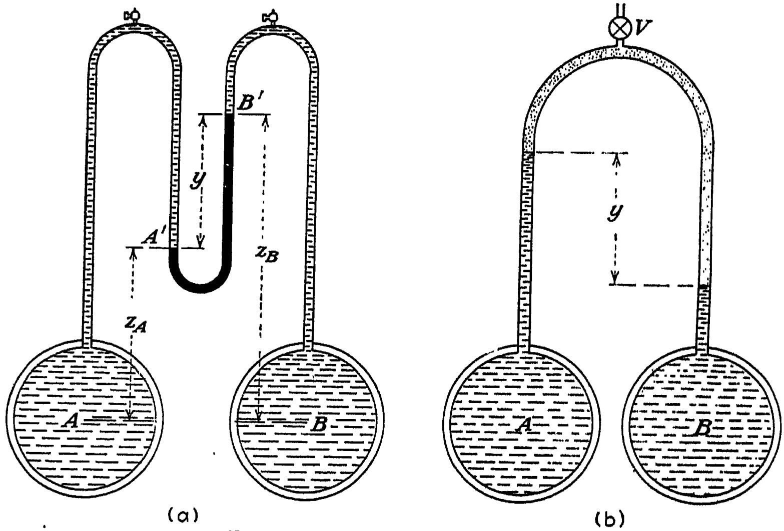

FIG. 16.—Differential manometers.

of specific weight w', the pressure at A' will exceed that at B' by the amount $w'y$, or $p_{A'} - p_{B'} = w'y$. But $p_{A'} = p_A - w''z_A$, and $p_{B'} = p_B - w''z_B$. Thus

$$p_A - p_B = w'y - w''(z_B - z_A) = (w' - w'')y.$$

To express this in terms of height of fluid of specific weight w gives

$$\frac{p_A}{w} - \frac{p_B}{w} = \left(\frac{w'}{w} - \frac{w''}{w}\right)y = (s - s'')y.$$

But if, as is the usual case, the temperature and hence specific weight of the fluid in the connecting piping is practically the same as that in vessels A and B, this reduces to

$$\frac{p_A}{w} - \frac{p_B}{w} = (s - 1)y.$$

In the differential manometer the left-hand column of mercury, or whatever it is, has a column of fluid of height y resting upon it which is not balanced by a corresponding amount on the right-hand column; hence the pressure difference is not sy alone.

It is obvious that if A and B are not at the same elevation, the pressures computed by the equation above must be corrected for the difference in level.

The differential manometer, when filled with a heavy liquid, such as mercury, is suitable for measuring large differences in pressure. For small pressure differences a light fluid, such as oil or even air, may be used, and in this case the manometer is arranged as shown in Fig. 16 (*b*). Naturally the fluid must be something that will not mix with the fluid in A or B. By the same method of analysis as the preceding it may be shown that

$$\frac{p_A}{w} - \frac{p_B}{w} = (s'' - s')y,$$

where s' is a specific gravity whose value is less than 1. In most cases the value of s'' is practically 1, so the expression reduces to

$$\frac{p_A}{w} - \frac{p_B}{w} = (1 - s')y.$$

As the density of the measuring fluid approaches that of the fluid being measured, the value of $(1 - s')$ approaches zero, and very high values of y are obtained for very small pressure differences. Thus the instrument becomes very sensitive.

For conditions intermediate between these two extremes it is often very satisfactory to use air instead of a liquid and to pump the air into the manometer in Fig. 16 (*b*) through valve V until the pressure is such as to bring the tops of the two water columns to the desired elevations. Any change in this air pressure raises and lowers both water columns by the same amount so that the difference between them is constant. In this case the value of s' may be considered as zero, since the density of air is relatively negligible, and the difference in pressure is given by y direct. For high pressures and small pressure differences, the value of s' is not negligible, however.

EXAMPLES

20. A mercury pressure gage (Fig. 13) is connected to a pipe line carrying water at 212°F. The temperature of the mercury and the water in the tubing is 60°F. If the elevation of point B is 12 ft. above A and the mercury reading is 60 in., what is the pressure in the pipe in pounds per square inch and in feet of water at 212° (see Table III)?

Ans. 34.6 lb. per sq. in., 83.3 ft.

21. A manometer such as is shown in Fig. 14 is used to measure a pressure in A of water at 100°F. The point B is 3 ft. above A. If the mercury and water in the tubing are at a temperature of 60°F. and the mercury reading is 10 in., what is the pressure in A (see Table III)? *Ans.* -8.37 ft.

22. Two vessels are connected to a differential manometer using mercury, the connecting tubing being filled with water. The higher pressure vessel is 6 ft. lower in elevation than the other. The temperature of both water and mercury is 50°F. If the mercury reading is 24 in., what is the pressure difference in feet of water? If another type were used employing a measuring liquid with a specific gravity of 0.8, what would be the reading for the same case? *Ans.* 31.14 ft., 125.7 ft.

23. The mercury and water in a differential manometer are both at a temperature of 80°F., while the water that is being measured is at a temperature of 39°F. If y = 5 ft. of Hg., find the pressure difference in pounds per square inch and feet of water at 80°. *Ans.* 27.1 lb. per sq. in., 62.7 ft.

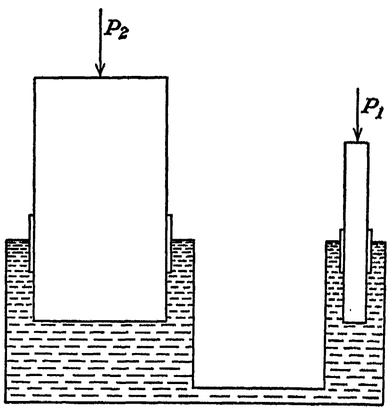

FIG. 17.—Hydraulic press.

19. The Hydraulic Press.—The most important device operating upon the principle of equal transmission of intensity of pressure in all directions is the hydraulic press. If in Fig. 17 a force P_1 be applied to the small piston whose area is A_1, the intensity of pressure throughout the whole volume of liquid will be increased by the amount $p = P_1/A_1$. Then the total additional force exerted upon the face of the large piston will be $pA_2 = (P_1/A_1)A_2 = P_1(A_2/A_1)$. It is thus seen that a small force exerted on the smaller piston is enabled to oppose a much greater load on the large piston. If G_1 and G_2 denote the weights of the pistons while z is the difference in elevation of their faces, the

result is, for equilibrium,

$$\frac{P_1 + G_1}{A_1} = \frac{P_2 + G_2}{A_2} \pm wz.$$

Since the volume of liquid in the vessel must remain constant, it follows that the distance moved by the larger piston must be much less than that moved through by the smaller one.

EXAMPLE

24. In Fig. 17 the diameter of the smaller piston is 1 in., and that of the larger one is 30 in. The big plunger weighs 10,000 lb. and sustains an external load of 15,000 lb. Assuming the difference in level of the two to be negligible and that the small piston weighs 10 lb., what total force P_1 is necessary to secure equilibrium? What distance will the small piston have to move to raise the other 0.2 in.? *Ans.* 17.8 lb., 15 ft.

20. PROBLEMS

25. A mercury U-tube contains mercury at a temperature of 50°F., which is also the temperature of the water in the connecting tube, while the temperature of the water whose pressure is being measured is 140°F. If the point B (Fig. 13) is 4 ft. above A and the mercury reading is 36 in., find the pressure in A in both pounds per square inch and feet of water at 140° (see Table II). *Ans.* 19.38 lb. per sq. in., 45.47 ft.

26. A differential manometer such as shown in Fig. 16 (*b*) contains a fluid with a specific gravity of 0.95 and gives a reading of 30 in. What is the corresponding pressure difference in feet of water? What would be the height of mercury in the type in Fig. 16 (*a*) for this same pressure difference? Assume temperature of 50°F. in every case.

Ans. 0.125 ft., 0.1195 in.

27. If the pressure of oil in A were to be measured by the gage shown in Fig. 13, and if the height of B were to be observed in a glass tube, it would be desirable to have a column of water above B to prevent the glass's being covered with oil. If the height of this column of water above B were x and its specific weight were w''', prove that $p_A = w''z + w'y + (w'' - w''')x$.

28. If a mercury gage, such as in Prob. 27, were used to measure the pressure of crude oil in a pipe line with a specific gravity of 0.85 at 60°/60°, and the oil, water, and mercury were all at a temperature of 60°F., the distance x were 36 in., z were 48 in., while the mercury reading was 24 in., what would be the oil pressure in pounds per square inch?

Ans. 13 lb. per sq. in.

29. The diameter of the tube C in Fig. 13 is d_1, while the diameter of tube B is d_2. Let z_0 be the elevation of the mercury when the two mercury columns are at the same height, while R is the reading of column C above

this zero level. Prove that

$$p_A = w''z_0 + \left(w' + (w' - w'')\left(\frac{d_1}{d_2}\right)^2\right)R$$
$$= M + NR,$$

where M and N are constants.

30. The specific gravity of a gas compared to air at 14.72 lb. per sq. in. and 70°F. is 0.55. The specific weight of the air is 0.0750 lb. per cu. ft. under these conditions. The average specific weights of both air and gas through a range of 1,000-ft. increase in altitude are 0.985 times the initial values. If the pressure in a gas main at a point where the atmospheric pressure is 14.72 lb. per sq. in. is 2 in. of water, what will it be at an elevation 1,000 ft. higher? *Ans.* 8.4 in. water.

CHAPTER III

HYDROSTATIC PRESSURE ON AREAS

21. Total Pressure on Plane Area.—Since fluids at rest are being dealt with, no tangential forces can be exerted and hence all forces are normal to the surfaces in question. Thus in hydrostatics there is no difference between perfect fluids and actual fluids, except for such effects as capillarity in liquids. If the pressure were uniformly distributed over an area, the total or resultant pressure would be the product of the area and the intensity of pressure, and the point of application of the force would be the center of gravity of the area. In general, the intensity of pressure is not uniform; hence further analysis is necessary.

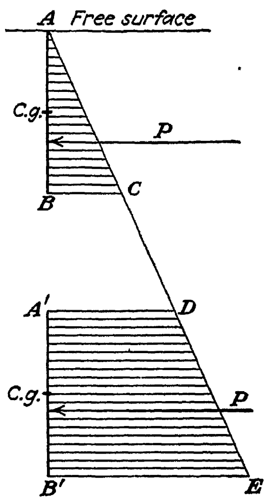

Fig. 18.

In Fig. 18 consider a vertical plane whose upper edge lies in the free surface. Let this plane be perpendicular to the plane of the paper so that AB is merely its trace. The intensity of pressure will vary from zero at A to BC at B. It will thus be seen that the total pressure P will be the summation of the products of the elementary areas and the intensities of pressure upon them. It is also apparent that the resultant of this system of parallel forces must be applied at a point below the center of gravity of the area, since the center of gravity of an area is the point of application of the resultant of a system of uniform parallel forces. If the plane be immersed to $A'B'$, the intensity of pressure varies from $A'D$ at A' to $B'E$ at B'. Since the proportionate change of intensity of pressure from A' to B' is less than before, it is clear that the center of pressure will approach nearer the center of gravity.

In Fig. 19 let MN be the trace of a plane making any angle θ with the horizontal. The view to the right is the projection of this area upon a vertical plane which is also normal to the plane containing MN. Let z be the depth of any point and y be the distance of any point from OX, the axis of intersection of the plane, produced if necessary, and the free surface. The coordi-

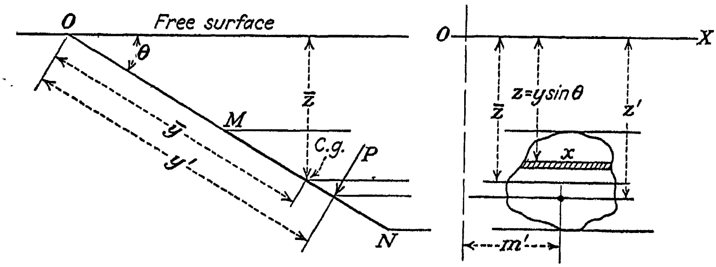

FIG. 19.

nates of the center of gravity of the area may be denoted by $\bar{z}$ and $\bar{y}$. The coordinates of the point of application of the resultant force P are z' and y'.

Consider an element of area dA so chosen that the intensity of pressure may be uniform over it. Such an elementary area may be represented by a horizontal strip across the plane. If x denotes the width of the plane at any point, then

$$dA = x\,dy,$$

and, since z is constant, the intensity of pressure is constant. Thus

$$p = wz,$$
$$dP = p\,dA = wz\,x\,dy.$$

Since

$$z = y \sin \theta,$$

then

$$P = w\int zx\,dy = w \sin \theta \int xy\,dy. \qquad (5)$$

If x can be expressed as an algebraic function of y, the preceding may be integrated, and the value of the total pressure in any given case may be found. Suppose, however, that the area is irregular in outline, as shown in Fig. 19, so that, while simultaneous values

of x and y may be found by measurement, the relationship between them cannot be expressed in any simple way. Or suppose there is an algebraic equation that applies but that the resulting expression is such that it cannot readily be integrated. In any practical case, a numerical value of the integral can be found by the following graphical means.

Multiply simultaneous values of x and y and plot the product xy against values of y, as is shown in Fig. 20. The area under this curve is the value of the integral.[1]

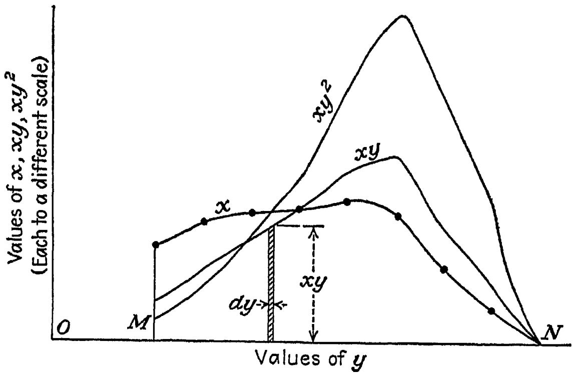

FIG. 20.

In case the area is a simple geometrical figure, as it usually is, so that its area and the location of its center of gravity can readily be computed, a simpler procedure than that just outlined may be followed. Since $x\ dy$ is the elementary area dA, Eq. (5) may be written

$$P = w \sin \theta \int y\ dA;$$

but $\int y\ dA$ is known to be $\bar{y}A$.

Hence

$$P = w \sin \theta\ \bar{y}A.$$

[1] An element of area of this figure may be seen to have a length xy and a width dy. Hence the summation of all such elementary areas must be the value of $\int (xy)dy$.

In general, if u and v are coordinates of an area, and u is plotted to such a scale that 1 in. = a units, while the scale for v is 1 in. = b units, then the scale for the area is 1 sq. in. = ab units.

It is not necessary actually to plot the area and measure it by a planimeter, as the value of the area can be computed by various methods, such as Simpson's rule or Durand's rule.

But $\bar{y} \sin \theta = \bar{z}$, which is the depth of the center of gravity of the area below the free surface. Therefore,

$$P = w\bar{z}A. \tag{6}$$

Thus the total pressure on any plane area may be found by multiplying the area by the depth of its *center of gravity* and by the specific weight of the liquid. The value of this quantity is independent of the angle of inclination of the plane.

EXAMPLES

31. A rectangular plane area is 5 by 6 ft., the 5-ft. side being horizontal and the 6-ft. side vertical. Determine the magnitude of the resultant pressure when the top edge is (*a*) in the water surface, (*b*) 1 ft. below the surface, (*c*) 100 ft. below the water surface.

Ans. (*a*) 5,620 lb., (*b*) 7,500 lb., (*c*) 193,000 lb.

32. Find the numerical value of $\int \sqrt{x^2 - 25}\, dx$ between the limits of $x = 6$ and $x = 20$ by graphical integration.

33. Assume an area that is at an angle of 30 deg. with the horizontal and the top point of which is at a vertical distance of 2 ft. below the water surface. Let the following be values of the horizontal widths in feet taken at intervals of 6 in. along the plane: 0, 2, 3, 3.4, 3.5, 3.6, 3.6, 3.6, 3.5, 3.4. Determine the magnitude of the total pressure on the area.

Solution: The initial value of y is 4 ft. and is multiplied by the first width, which happens to be 0 in this case, though a finite value might just as well appear, depending upon the shape of the area. Doing the same for each width, the following values of xy are obtained: 0, 9, 15, 18.7, 21, 23.4, 25.2, 27, 28, 28.9. These values may be plotted against corresponding values of y, and the area under the curve will be the value of $\int xy\, dy$ or $\bar{y}A$. Note that the area just obtained is the value of the *moment* of the plane area with which the problem is concerned with respect to the axis O of Fig. 19.

Solving by Simpson's rule for the sake of illustration, though, unlike Durand's, it may be used only with an odd number of ordinates and hence adds one more complication here,

$$(0 + 28) + 4(9 + 18.7 + 23.4 + 27) + 2(15 + 21 + 25.2) = 462.8.$$

This is seen to be obtained by multiplying even-numbered ordinates by 4 and odd ones by 2, except the first and the last. This sum should then be multiplied by one-third of the interval, which is $\frac{1}{6}$ ft. The result is 77.1. To this must be added the trapezoidal area left over, whose width is 12 ft. and whose two bases are 28 and 28.9. This area is 14.2, giving a total area whose value is 91.3 cu. ft., since these are the units here involved. Hence the total pressure is

$$P = 62.4 \times 0.500 \times 91.3 = 2,850 \text{ lb.}$$

34. Find the area and the location of the center of gravity of the plane figure in the preceding problem. *Ans.* 14.1 sq. ft., $\bar{y} = 6.48$ ft.

22. Depth of the Center of Pressure.—The point of application of the resultant force on an area is called the center of pressure. The line of action of a force is usually located by taking moments. In this case it is convenient to take OX in Fig. 19 as the axis of moments. On any element of area dA the total pressure is

$$dP = w \sin \theta \, xy \, dy,$$

and its moment is

$$y \, dP = w \sin \theta \, xy^2 \, dy.$$

The moment of the resultant force being the sum of the moments of its components, if y' is the distance to the center of pressure,

$$y'P = w \sin \theta \int xy^2 \, dy.$$

As in the preceding article, this integral may be evaluated in a numerical case by finding the value of an area whose ordinates are values of xy^2. The analytical solution, on the other hand, gives the following: Since $x \, dy = dA$ and the integral of $y^2 \, dA$ is the moment of inertia of the area dA about the axis O,

$$y'P = w \sin \theta \, I_o.$$

This may be divided by the value of P in the preceding article, with the result that

$$y' = \frac{w \sin \theta \, I_o}{w \sin \theta \, \bar{y} A} = \frac{I_o}{\bar{y} A}. \qquad (7)$$

That is, the distance of the *center of pressure* from the axis, where the plane, or plane produced, intersects the water surface, is obtained by dividing the moment of inertia by the static moment of the area A about the same axis.[1]

[1] In Fig. 20 the area under the curve of values of x is equivalent to the actual area A of the plane surface in question, though it may be altered slightly in appearance, since one side of the actual area becomes here the straight line MN. If OM here represents the same distance as in Fig. 19, then the distance from the vertical axis through O to the center of gravity of the area under the x curve is the value of $\bar{y}$ for the area A. And the distance from this axis to the center of gravity of the area under the xy curve is the value of y' for the area A. Since each area is the moment of the area of lower degree, it follows that the value of y' may be obtained by dividing the area under the xy^2 curve by that under the xy curve, while the value of $\bar{y}$ may be obtained by dividing the area under the xy curve by that under the x curve, paying due regard to the scale values used in plotting.

This may also be expressed in another form, by noting that $I_o = \bar{y}^2A + I_g$, where I_g is the moment of inertia of an area about its own gravity axis, and that I_g in turn is equal to $k_g{}^2A$, where k_g is the radius of gyration about the gravity axis; hence

$$y' = \frac{\bar{y}^2A + k_g{}^2A}{\bar{y}A} = \bar{y} + \frac{k_g{}^2}{\bar{y}}. \tag{8}$$

From these equations it may be seen that the location of the center of pressure is independent of the angle θ; that is, the plane area may be rotated about the axis OX without affecting the location of the center of pressure. However, this will not hold for $\theta =$ *zero*, since the value of P would also be zero.

From Eq. (8) it may also be seen that the center of pressure is always below the center of gravity. Also as the depth of immersion is increased for a given value of θ, the distance $\bar{y}$ increases. But as k_g remains constant in value, it may be seen that the last term in Eq. (8) becomes relatively small; hence y' approaches $\bar{y}$ in value. The same thing would be true if the depth of the center of gravity $\bar{z}$ remained constant while the plane was rotated so as to approach a horizontal direction. (This is entirely different from rotation about the axis OX, since $\bar{y}$ no longer remains constant.)

EXAMPLES

35. Determine the depth of the center of pressure for the three cases in Prob. 31. *Ans.* 4.00, 4.75, 103.03 ft.

36. Determine the center of pressure for the case in Prob. 33, using Simpson's rule. *Ans.* $y' = 6.68$ ft.

23. Lateral Location of Center of Pressure.—For most practical problems the depth of the center of pressure is all that requires solution, since the areas dealt with are usually such that a straight line can be drawn through the centers of all horizontal lines. In such cases the center of pressure is seen to lie on this line. But in case this is not so, it would be necessary to compute m' as in Fig. 19, m' being measured from any axis parallel to trace MN.

Again moments, as in the preceding article, are employed. If m is the distance to the center of gravity of an element from the axis in question, the moment of dP is

$$m\,dP = wmy \sin\theta\, dA.$$

Hence the value of m' is

$$m' = \frac{\int m\, dP}{\int dP} = \frac{w \sin\theta \int my\, dA}{w \sin\theta \int y\, dA} \qquad (9)$$
$$dA$$

This equation differs from Eq. (7) simply because $\int my\, dA$ is given instead of $\int y^2\, dA$. The latter quantity is more frequently met with; it is given a name, symbolized by the letter I, and values of I for different areas can usually be found in tables. The former expression is called *product of inertia* and is symbolized by the letter J; but owing to the infrequent use that is made of it, values of J cannot usually be obtained save by integration. Lacking the knowledge of the value of J for any area, simply proceed to evaluate $\int my\, dA$, just as $\int y^2\, dA$ should be evaluated in case the value of I for the area in question were not known.

It will be found that reduction formulas can be used here as with moments of inertia. If J indicates the product of inertia with respect to the intersection of any two axes, while a and b are the coordinates of the center of gravity of an area about which the product of inertia is J_g, it will be found that

$$J = J_g + Aab.$$

In using Eq. (9) it must be noted that y is to be measured as in Fig. 19, while m may be measured from any axis in the plane of the figure and perpendicular to OX.

EXAMPLES

37. Given a right triangle with height h and base b, with its vertex in the water surface and its plane vertical. Find the value of y' and then determine m': (*a*) by inspection; (*b*) by calculus. *Ans.* $y' = \frac{3}{4}h$; $m' = \frac{3}{8}b$.

38. Find the center of pressure on an area which is a quadrant of a circle. It is placed in a vertical plane, and one edge lies in the water surface.
Ans. $\bar{y} = 4r/3\pi$; $y' = 3\pi r/16$; $m' = 3r/8$.

24. Resultant Thrust on Plane Areas.—So far, the total pressures on one side of a plane area alone have been dealt with. Of course, when the area is completely immersed in a fluid as shown in some of the previous illustrations, the total pressure on one side is balanced by that on the other, and the net effect is zero. But when the two sides are not subjected to the same pressure, there is a resultant thrust whose value is desired.

So far, the surface of the liquid has been considered as being free from all pressure. Thus, in Fig. 21 the intensity of pressure should be considered as varying from zero at A to BC at B. But in reality there is some pressure, in general, from the atmosphere acting upon the water surface equivalent to a height of about 34 ft. of water, and thus the true free surface might really be at point O, the distance AO being equal to the height of the water barometer. The absolute intensity of pressure upon the left-hand side of the plane AB, therefore, varies from AD to BE. But in practical applications the difference between the pressure on the left-hand side and that on the right-hand side is desired. But the pressure on the right-hand side is that due to the atmosphere, and its intensity is uniform from A to B, being equal to AD'. But $AD' = AD = CE$. Hence atmospheric pressure is added alike to both sides, and it is useless to consider it. Therefore, atmospheric pressure is neglected altogether, and the water surface is treated as a true free surface in most calculations.

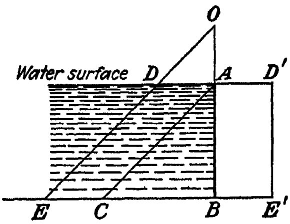

FIG. 21.

Suppose that there is an area, such as AB in Fig. 22, with a fluid pressure on both sides but of different intensities. Of course, the magnitudes of the total pressures on both sides of the area could be computed, and the difference would be the resultant desired. But it would also be necessary to find the centers of pressure on both sides and then locate the line of action of the resultant of these two forces. The following analysis will indicate a much easier solution.

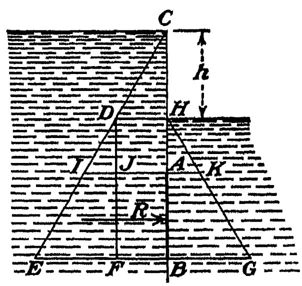

FIG. 22.

At A the intensities of pressure on the two sides are AI and AK. If IJ be laid off equal to AK, the net difference in the intensity of pressure will be AJ. In similar manner at B the net intensity of pressure is BF. And it is readily seen that, since CDE and HKG make the same angle with the vertical, the values of HD, AJ, and BF are equal. Thus the resultant inten-

sity of pressure on the area AB is uniform and equal to HD in value. But HD is the intensity of pressure at the depth h. Hence the resultant thrust on any area with both sides completely covered by the same liquid is

$$R = whA, \tag{10}$$

where h is the difference in level of the two liquids. And since the net intensity of pressure is uniform, the resultant thrust will act through the center of gravity of the plane area.

25. Horizontal Pressure on Curved Surface.—On any curved or irregular area in general, such as that whose trace is AB in Fig. 23, the pressures upon different elements are different in

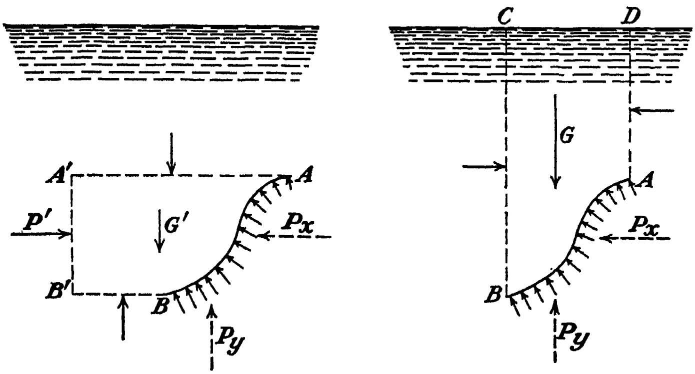

FIG. 23.

direction, and an algebraic or calculus summation is impossible. Hence Eq. (6) can be applied only to a plane area. But the component of pressure in certain directions may be found. Thus if each dP is multiplied by $\cos\theta$, θ being a variable angle which each elementary force makes with the horizontal, the total horizontal force would be

$$P_x = \int dP \cos\theta. \tag{11}$$

In general, it will be tedious to integrate the latter and often practically impossible. Hence the following procedure may be employed.

Project the irregular area in question upon a vertical plane, the trace of the latter being $A'B'$. The projecting elements are AA', BB', etc. It is seen that these projecting elements, which are all horizontal, enclose a volume whose ends are the vertical

plane $A'B'$ and the irregular area whose trace is AB. This volume of liquid is in equilibrium under the action of the following forces. Upon the vertical plane at the left there is a force P'; gravity G' acts upon the volume and is vertical; the pressures on the projecting elements are all normal to these elements, hence normal to P'. Then there are the pressures upon the area in question at the right-hand end, the horizontal component of pressure being represented by P_x and the vertical component by P_y. Since a condition of equilibrium exists, the sum of all the forces in any direction must be equal to zero. But in a horizontal direction the only forces are P' and P_x. Hence

$$P_x = P'. \tag{12}$$

That is, the component, in any given horizontal direction, of the pressure upon any area whatever is equal to the pressure upon the projection of the area upon a vertical plane which is perpendicular to the given horizontal direction. The lines of action must also be the same.

26. Vertical Pressure on Curved Surface.—The vertical component of pressure on an irregular surface can be found by a method similar to that for the horizontal pressure. Thus in Fig. 23 if a volume of liquid is taken, of which the area in question forms the base and vertical elements such as AD and BC form the sides, it is found that the following forces are acting. Considering CD a free surface, the pressure on the upper face is zero. The pressure on the lower face is composed of the two components P_x and P_y. Gravity, G, is the only other vertical force, the pressures on the sides all being horizontal. Summing up the vertical forces and equating to zero,

$$P_y = G. \tag{13}$$

Hence the vertical component of pressure on any area whatever is equal to the weight of that volume of liquid which would extend vertically from the area to the free surface.

27. Component of Pressure in Any Direction.—In general, the component of pressure in any direction aside from horizontal and vertical cannot be found, since the weight of the volume of liquid, such as $AA'B'B$ in Fig. 23, would have to enter the equation. But if the depth of immersion is great so that the pressures on AB and $A'B'$ are great compared with the weight G', the latter

may be neglected. Hence, in such cases only, the component of pressure in any direction may be taken as the pressure upon an area projected in that direction upon a plane that is perpendicular to the given direction.

Of course with a plane area the component of pressure in any direction may be found by multiplying P by the proper function of some angle. Or it may be convenient to find it by the methods of Arts. 25 and 26. Also for a plane area, since $P \cos \theta = (w\bar{z}A) \cos \theta$, it may be seen that the component of pressure is the same as the pressure upon an area of value $A \cos \theta$, provided the center of gravity of such area be the same depth as the center of gravity of the given plane.

28. Resultant Pressure on Curved Surface.—In general, there is no single resultant pressure on an irregular surface, for a system of nonparallel and noncoplanar forces does not usually reduce to

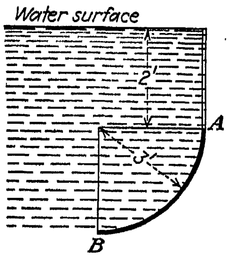

FIG. 24.

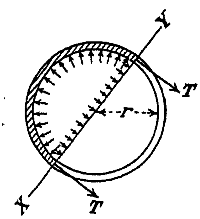

FIG. 25.

anything simpler than two single forces. Thus, in general, P_x and P_y are not in the same plane and hence cannot be combined. But in some special cases of symmetrical surfaces, these two components will lie in the same plane and hence can be combined into a single force.

EXAMPLE

39. In Fig. 24 is shown a quadrant of a circular cylinder AB, whose length perpendicular to the plane of the paper is 4 ft. (*a*) Find the horizontal component of pressure. (*b*) Find the vertical component of pressure. (*c*) Find the magnitude and direction of the resultant water pressure. (*d*) What locates its line of action?

Ans. (*a*) 2,622 lb., (*b*) 3,262 lb., (*d*) $z' = 3.7$ ft., $m' = 1.38$ ft. from B.

29. Pipes under Internal Pressure.—If the internal pressure in a cylindrical pipe is great enough to be considered in determining the thickness of pipe wall necessary, it will be large enough so

that the weight of the water may be disregarded. Hence according to Art. 27 we may compute the resultant pressure in any direction. Suppose that in Fig. 25 we pass a plane XY through a diameter of the pipe as shown. The total pressure on one-half of the pipe in any direction, such as that normal to XY, will evidently be $p \times 2r \times l$, l being any length of pipe. This follows directly from Art. 27 or may be seen from the fact that the thrust of the water on the wall of the pipe normal to XY must be balanced by the thrust of the water on the plane XY. This pressure will tend to rupture the pipe across the plane XY and is resisted by the tensions in the walls of the pipe, such as T. Evidently $2T = 2prl$. If the thickness of the pipe wall be denoted by t, and the stress induced in it by S then $T = Stl$. Hence

$$St = pr. \tag{14}$$

From Eq. (14) the thickness of wall necessary may be computed for any allowable unit tensile stress. However, it is well to note that p should be the maximum intensity of pressure that may occur, and in case of water hammer these intensities are much greater than the static pressures alone. Also, it may often be found that Eq. (14) gives entirely too thin a wall to stand ordinary handling and to allow for a certain amount of corrosion. In practice p is, therefore, increased to allow for possible water hammer, and the thickness determined by Eq. (14) is then increased to a value necessary for these other reasons. The tension in the case shown is called *hoop tension.*

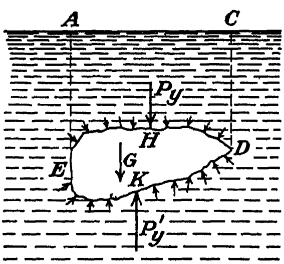

FIG. 26.

For cylinders with thin walls this formula will hold, since it assumes uniform intensity of stress across the metal. But with thick walls it does not hold. In the case of hoop tension in a cylinder with thick walls it is usually assumed that the intensity of stress is a maximum at the inner face and decreases to zero at the outside of the wall. Also, the elasticity of the material enters into the hypothesis. The various formulas proposed are largely empirical and will not be given here.

30. Buoyant Force of the Water and Flotation.—Considering the body $EHDK$ immersed in a fluid in Fig. 26, it is seen that it is

acted upon by gravity and the pressures from the surrounding fluid at least. In addition there may be other forces applied. On the upper surface of the body the vertical component of the pressure P_y will be equal to the weight of the volume of fluid $AEHDC$. In similar manner, the vertical component of the pressure on the under surface P'_y will be equal to the weight of the volume of fluid $AEKDC$. It is evident that P'_y is greater than P_y and that the total vertical force exerted by the fluid is upward and is equal in magnitude to

$$P'_y - P_y = \text{weight of volume } AEKDC - \text{weight of volume } AEHDC.$$

But the difference between these two volumes is the volume of the body $EHDK$. Hence for any body immersed in a fluid such as water the buoyant force of the water is equal to the weight of the water displaced.

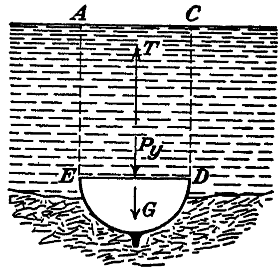

FIG. 27.

If the body remains in equilibrium in the position shown in Fig. 26, when no other forces are acting, it is seen that $G = P'_y - P_y$. Hence the body must be of the same density as the fluid in which it is immersed. If it is lighter than the fluid, a downward force will have to be applied whose value is $B - G$, B being the buoyant force of the fluid. If the body is denser than the fluid, it will have to be supported by a force whose value is $G - B$. But if the body rests on the bottom of a body of fluid (Fig. 27) in such a way that the fluid does not have access to the underside, there will be no buoyant effect, for then P'_y = zero. Thus in the case of a ship, for example, sunk in the mud at the bottom of a body of water, the pull T necessary to raise the ship is not only the weight of the ship but also the weight of the entire volume of water resting on top of it. Thus, in Fig. 27, $T = G + P_y$.

If no external forces are applied to a body that is lighter than the fluid, it will float on the surface, such portion of its volume being immersed as is necessary to displace an amount of fluid equal in weight to the weight of the body.

If the body is slightly heavier than the fluid, it will sink. If it

is less compressible than the fluid and there is sufficient depth, it will sink until such a depth is reached that the density of the fluid is equal to its own density. If it is more compressible than the fluid, its own density will be increased more rapidly than that of the water, and it will sink to the bottom.

EXAMPLES

40. An iceberg floats with one-seventh of its volume above the surface. What is its specific gravity relative to ocean water? What portion of its volume would be above the surface if it were in fresh water?

Ans. $6/7$, 0.12.

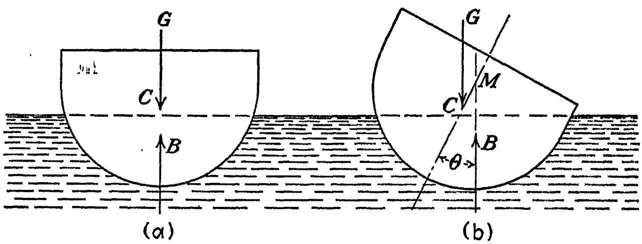

FIG. 28.

41. A balloon weighs 250 lb. and has a volume of 10,000 cu. ft. When filled with hydrogen which weighs 0.0056 lb. per cu. ft., what load will it support in air weighing 0.07 lb. per cu. ft.? *Ans.* 394 lb.

42. A body whose volume is 3 cu. ft. weighs 320 lb. What force will be necessary to sustain it (*a*) when it is immersed in fresh water; (*b*) in ocean water? *Ans.* (*a*) 132.8 lb.; (*b*) 128 lb.

31. Metacenter.—For a body floating on the surface of the water, such as in Fig. 28, there are only the two vertical forces, its weight G and the buoyant force of the water B. The latter acts through the center of gravity of the water displaced. This point is called the center of buoyancy. If the body is in equilibrium, these two forces must be in the same straight line. Suppose that by some external agency the body is rolled or displaced through some angle θ. The center of gravity is naturally unchanged in its position in the section, but the center of buoyancy, in general, will change. Thus G and B constitute a couple. In Fig. 28 (*b*) this is a righting couple, since it tends to restore the body to the upright position.

It may be seen that the line of action of B cuts the axis at point M. This point is called a *metacenter*. As the angle θ varies, the amount of this couple will vary, and the point M will

also change its location. The position that M approaches as θ approaches zero is the *true metacenter*. It may be seen that if the couple is a righting couple the point M must always be above C the center of gravity. It is necessary in ship design to insure that M will be above the center of gravity for all angles of heel. Thus it is necessary not only to locate the true metacenter but also to compute the moment of the righting couple for all values of θ which are likely to be encountered. Further consideration of this topic properly belongs to the subject of ship design.

32. PROBLEMS

43. Suppose that a vertical rectangular gate is 3 ft. wide by 4.5 ft. high, that water is on one side only, and that the upper edge lies in the water surface. If the sole support is a shaft attached along the bottom edge, what twisting moment must be applied to the shaft to hold the gate against the water pressure? *Ans.* 2,840 ft.-lb.

44. Suppose that the rectangular gate described in Prob. 43 has its upper edge 1.25 ft. below the water surface; what twisting moment must be applied to the shaft? *Ans.* 5,220 ft.-lb.

45. Suppose that the supporting shaft is in the upper edge of the gate described in Probs. 43 and 44; what would be the required moment in each case? *Ans.* 5,680 ft.-lb., 8,060 ft.-lb.

46. At what depth below the surface would the top edge of the gate described in Prob. 43 have to be in order that the center of pressure would be 0.2 ft. below the center of gravity? *Ans.* 6.175 ft.

47. Suppose that Fig. 19 represents a rectangular area 5 by 6 ft.; that $MN = 6$ ft., the 5-ft. edge being normal to the plane of the paper; and that $\bar{y} = 4$ ft. Find the magnitude of the total pressure and the location of the center of pressure when θ has values of 90, 60, 30, and 10 deg. *Ans.* (*a*) $P = 7{,}488$ lb., $y' = 4.75$ ft.; (*b*) $P = 6{,}490$ lb.; (*c*) $P = 3{,}744$ lb.; (*d*) $P = 1{,}302$ lb.

48. Suppose that in Prob. 47 $\bar{y}$ was variable but that $\bar{z} = 4$ ft. Solve with values of θ of 90, 60, 30, and 0 deg. *Ans.* (*a*) $P = 7{,}488$ lb., $y' = 4.75$ ft.; (*b*) $y' = 5.265$ ft.; (*c*) $y' = 8.375$ ft.; (*d*) $y' =$ infinity, $z' = 4$ ft.

49. Find the depth of the center of pressure on a vertical triangular area whose altitude is h and whose base is b, if (*a*) its vertex lies in the water surface, and its base is horizontal; (*b*) its base lies in the water surface. *Ans.* (*a*) $y' = \frac{3}{4}h$; (*b*) $y' = \frac{1}{2}h$.

50. Suppose that a cylinder 1 ft. in diameter and 3 ft. high is filled with water to the top. Find the total pressure on the bottom by computing the weight of the water and then by using Eq. (6). Then assume the top of the cylinder to be covered except for a small pipe which extends vertically for some distance. Assume the diameter of this pipe to be so small that a pint of water poured into it fills it with water to a height of 20 ft. above the top of the cylinder. Find the total pressure on the bottom. *Ans.* 147 lb., 1,128 lb.

51. A vertical plane area, whose upper edge coincides with the water surface, has the following widths starting with the surface and at intervals of 1 ft. below it: 4.90, 4.48, 4.00, 3.46, 2.82, 2.00, and 0 ft. Plot values of x, zx, and z^2x and determine the magnitude of the resultant pressure, the depth of the center of gravity, and the depth of the center of pressure.

Ans. 2,930 lb., 2.40 ft., 3.43 ft.

52. A vertical plate 8 ft. high and 10 ft. wide is immersed with the upper edge lying in the water surface, but there is a 4-ft. square removed from the upper right-hand corner. Find the total pressure and also locations of center of gravity and center of pressure vertically and laterally from the left-hand edge.

Ans. P = 17,960 lb., $\bar{z}$ = 4.5 ft., $\bar{m} = m'$ = 4.25 ft., y' = 5.6 ft.

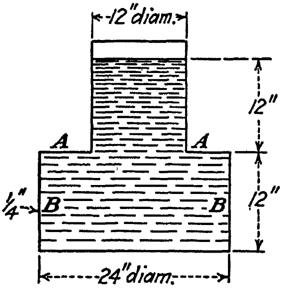

FIG. 29.

53. Figure 29 shows a cylindrical tank. What is the total pressure on the bottom? What is the total pressure on the annular surface A-A? Find the maximum intensity of longitudinal tensile stress in side walls B-B; (*a*) if the tank is suspended from the top; (*b*) if it is supported on the bottom. *Ans.* 392 lb., 147 lb.; (*a*) 20.8 lb. per sq. in.; (*b*) 7.8 lb. per sq. in.

54. A vertical equilateral triangle has its vertex in the water surface and its base horizontal. If the sides are 9 ft. long, find total pressure and location of center of pressure.

55. A vertical rectangular area is 9 ft. high and 6 ft. wide, and from its upper corner a section 3 ft. square is removed. The upper edge is 2 ft. below the water surface. Find the magnitude of the total pressure and the coordinates of the center of pressure.

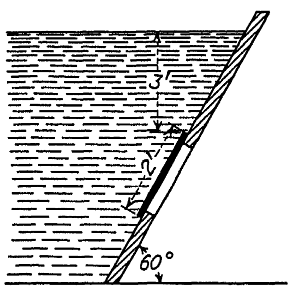

FIG. 30.

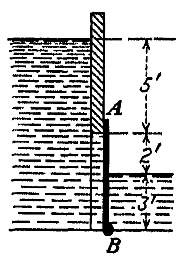

FIG. 31.

56. A rectangular wrought-iron caisson is to be sunk in 20 ft. of water. It is in the form of a box 50 by 20 by 23 ft. high and weighs 75 tons. How deep will it sink when launched? What weight must be added to cause it to sink to the bottom? *Ans.* 2.4 ft., 550 tons.

57. A plank 10 ft. long of uniform cross section and weighing 29 lb. per cu. ft. is hinged at a point 2 ft. from one end. The other end dips into the water. Find the length of the submerged portion.

58. Find the magnitude and point of application of the resultant pressure on the 2-ft. circular gate shown in Fig. 30. *Ans.* 758 lb., 4.52 ft.

59. The gate *AB* in Fig. 31 rotates about an axis through *B*. If the width is 4 ft., what torque applied to the shaft through *B* is required to keep the gate shut? *Ans.* 19,677 ft.-lb.

CHAPTER IV

DAMS

33. The Gravity Dam.—One of the most important applications of hydrostatics is to the design of dams, of which there are several types. The gravity dam is one that depends for its stability upon its weight. A typical section of such a dam is shown in Fig. 32, which is drawn to scale, as are also the force vectors. Unless the face *AB* is vertical, it will be necessary to

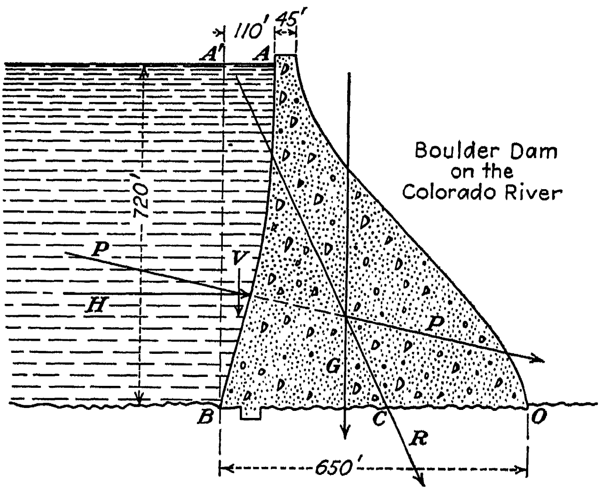

FIG. 32.—Cross section of gravity dam.

compute the two components of the water pressure, *H* being equal to the pressure on a plane whose trace is *A′B*, while *V* is the weight of the volume of water represented by *ABA′*. Since the stability of a gravity dam is independent of the total length, it is customary to base all calculations upon a section 1 ft. long (perpendicular to the plane of the figure).

The total water pressure *P* combined with the weight of the section *G* gives a resultant pressure on the base whose value is *R*. This pressure is distributed over the base but may be considered to be concentrated at a single point of application *C*. If

R be resolved into its two components at C, the value of the horizontal component must be equal to H, while that of the vertical component will be equal to $G + V$. By taking moments of all the forces about O it will be easy to locate the point C.

If the dam rests tightly upon impervious rock, and there is no leakage of water along any plane, or if a cutoff wall at B runs down deep enough to stop percolation, and the base of the dam is well drained, the preceding forces are all that act upon the dam, excepting the supporting force of the earth which is equal and opposite to R. *But* if water does have access to the underside of the dam, a vertical upward pressure will be exerted upon BO, and R must then include this force also.

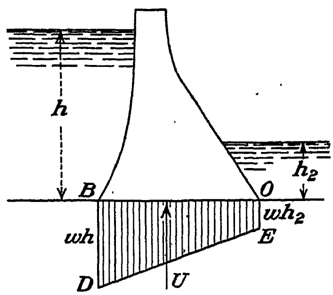

Fig. 33.—Water pressure on base.

How much this uplift may amount to depends upon conditions, and, as these are not definitely known, it is necessary to make assumptions. The most extreme case would be if water saturated the entire foundation but could not readily escape past O, so that the whole base would be subjected to a pressure equal to the height of the water in the reservoir. But such a case is quite improbable. In reality, the worst condition would more likely be such as is shown in Fig. 33. In case no water stood on the downstream side of the dam, the pressure diagram would reduce to a triangle. But since it is hardly possible for the dam to rest upon a film of water over its entire base, it is customary to assume that water pressure acts upon about one-half or at most two-thirds of the area, while the intensity of the pressure may vary from heel B to toe O, as shown in Fig. 33. Thus if K be this fraction and L the distance BO, the total uplift will be

$$U = \frac{KLw(h + h_2)}{2}.$$

Owing to the use of drainage wells, a modification of the preceding pressure variation is shown in Fig. 35, but even this proved to be conservative, as measurements made after the dam was completed showed that the pressure dropped to zero at the drain-

age wells. In other cases still other assumptions might be made to fit the hypothetical conditions.

It may be seen that the horizontal thrust of the water H may be opposed by friction between the dam and the foundation upon which it rests. If the coefficient of friction be denoted by μ, then if the dam is to be safe against sliding, the value of H must be less than $\mu(G + V - U)$. Actually, portions of the dam may be set into cutoff trenches, and usually the bedrock upon which the dam may rest is so irregular that sliding is out of the question. In the case of the dam shown in Fig. 35, successive pourings of the concrete were made with the pouring joints sloping so that

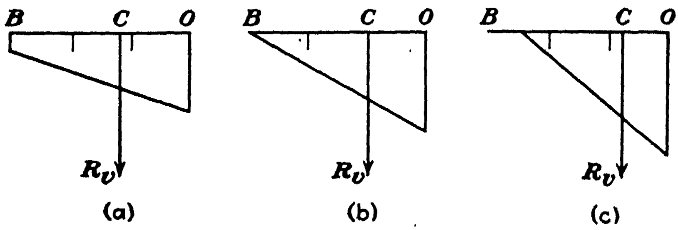

FIG. 34.—Load distribution on base.

if there were any tendency of any upper portion of the dam to slide at such a joint, the mass would have to move uphill.

If it were possible for the dam to act as a rigid body under all circumstances, and if the point C in Fig. 32 were to the right of O, the dam could then fail by overturning about the toe at O, as an axis. It is seen that H and U tend to overturn the dam but are resisted by G and V. The factor of safety against overturning is the ratio of the sum of the moments about O of G and V to the sum of the moments of H and U.

However, before a dam of large size would overturn, the material along the base near the toe would be crushed owing to the high pressure. Thus, although the point C might be to the left of O in Fig. 32 so that the structure would be safe against overturning, the base might still not be safe against crushing. Hence, it is necessary to consider the variation of the intensity of pressure along the base BO.

If the resultant pressure, whose vertical component is R_v, were applied at the center of the base, the intensity of pressure would be uniform over the base. But if the point C were as shown in Fig. 34 (a), the intensity of pressure would vary from B to O;

and if C were to be two-thirds the distance from B to O, the intensity of pressure would vary from zero at B to a value at O which would be just twice the average pressure. And if C were more than two-thirds the distance from B to O, the pressure at O would be still greater, while there would be no pressure at all for some distance from B. This conclusion rests upon the assumption that the deformations of both the dam and the

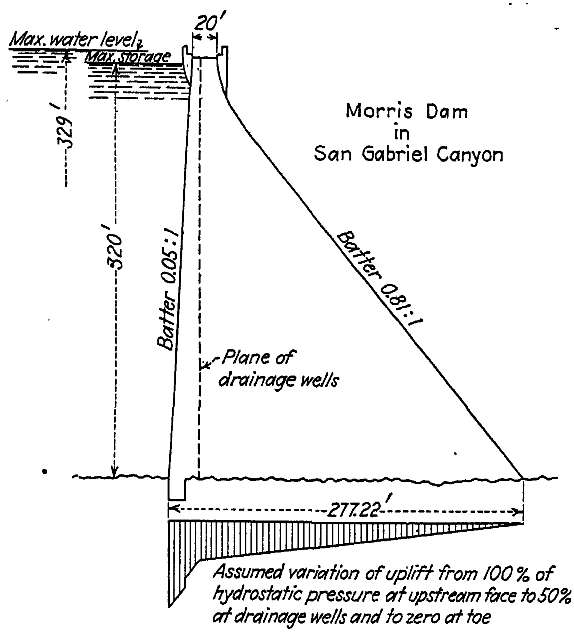

FIG. 35.—Cross section of gravity dam designed to withstand an earthquake of intensity 0.1 g horizontally.

foundation are linear. Actually, concrete can sustain some tension, and elastic deformation will vary somewhat from the linear and thus alter slightly the stress distributions assumed here. But since tensile stresses are to be avoided in concrete or masonry structures, and since it is desirable that there be compression at all points in the face to prevent any cracks or joints from opening up and admitting water, it is customary to design a dam with such proportions that the total resultant pressure cuts the base within the middle third.

As the water level in the reservoir lowers, the resultant pressure will cut the base at a point farther from the toe, and, when the

reservoir is empty, the resultant pressure is merely G, the weight of the dam. It is common practice to have the line of action of G also within the middle third, so that the pressure at the heel is not excessive and so that no tensile stresses can exist whether the reservoir is full or empty. For this reason the upstream face is usually given some batter, as shown in both Figs. 32 and 35.

It is not only necessary to analyze the dam as a whole but necessary also to investigate all portions of it with respect to any horizontal plane as a base, both for the maximum height of water and with the reservoir empty.

TABLE IV.—STRESS ANALYSIS OF MORRIS DAM[1]

Height above base, ft.	Upstream face vertical stresses, lb. per sq. in.			Downstream face					
				Vertical stresses, lb. per sq. in.			Stress parallel to face, lb. per sq. in.		
	329 ft.	320 ft. + earthquake	Empty	329 ft.	320 ft. + earthquake	Empty	329 ft.	320 ft. + earthquake	Empty
320	2.6	6.4	7.2	10.2	9.9	9.2	10.2	9.9	9.2
310	1.6	8.7	18.4	26.6	24.2	18.3	27.7	25.2	19.1
300	1.6	7.7	31.4	38.3	36.5	20.0	43.6	41.5	22.7
290	2.0	6.4	45.4	46.0	45.4	16.5	59.1	58.3	21.2
250	10.6	6.2	88.3	58.2	65.4	2.5	96.4	108.2	4.1
200	18.5	4.6	132.9	85.7	101.9	3.7	141.9	168.7	6.1
100	30.0	3.0	226.1	152.2	181.9	10.5	252.0	301.3	17.4
0	48.2	4.2	322.0	215.6	261.5	17.2	357.0	433.1	28.5

[1] Courtesy of S. B. Morris, chief engineer, Pasadena, Calif., Water Department.

Some results of such an analysis of the dam shown in Fig. 35 are given in Table IV. Stresses were computed for three cases: (*a*) with the water surface 9 ft. above the crest of the spillway, a height that would be reached only at rare intervals with a flood of unusual magnitude; (*b*) with the water surface level with the crest of the spillway, which would be the maximum storage level, and with the added effect of an earthquake of an intensity equal to 0.1 g; (*c*) with the reservoir empty. It may be seen that the effect of an earthquake is to increase the stress at the base on the downstream side and to decrease it on the upstream side.[1]

[1] WESTERGAARD, H. M., "Water Pressures on Dams during Earthquakes," *Proc. Amer. Soc. Civil Eng.*, vol. 57, no. 9, p. 1303, Nov., 1931.

34. The Framed Dam.—Contrasted with the gravity dam is the framed dam shown in Fig. 36, which depends for its stability upon the strength of its members. It consists of a watertight deck AB, supported by struts, trusswork, or buttresses at certain intervals along the length of the dam (perpendicular to the plane

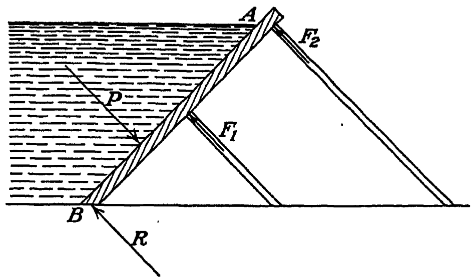

FIG. 36.—Framed dam.

of the figure). The deck is always inclined so that the weight of the water upon it may hold the structure down and increase the factor of safety against sliding.

35. The Arch Dam.—In the case of a short high dam in a situation where firm support can be had from the walls on either side, the arch dam is desirable. It is designed to withstand the water pressure by pure arch action and to transmit the pressures to the abutments at either end. The material in an arch dam is usually much less than in a pure gravity dam, but any arch dam

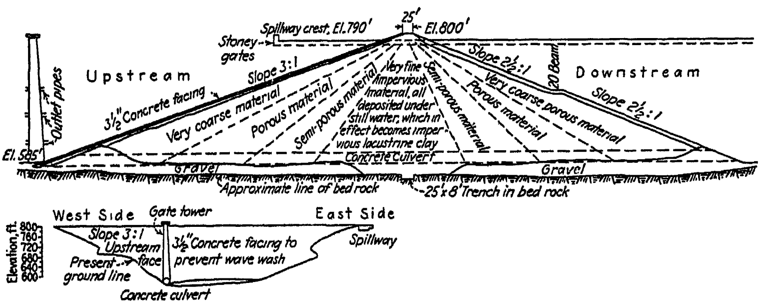

FIG. 37.—Section of original Calaveras earth dam.

acts to some extent as a gravity dam. Its analysis is not within the scope of this text.

36. The Earth Dam.—Under favorable circumstances the earth dam is a very economical type. A typical section of such a dam may be seen in Fig. 37. The slopes on both the upstream

and downstream faces are less than the angle of repose of the material used. In order to make such a dam watertight it is provided with an impermeable core which may be a thin vertical wall of concrete or other material.

FIG. 38.—Model of spillway for Morris dam with a discharge proportional to 40,000 cu. ft. per sec. (*Courtesy of S. B. Morris.*)

It was intended in constructing the dam shown in Fig. 37, which is the highest earth dam in the world, to obtain such a core by depositing fine material under water; but during the construction of the dam the hydrostatic pressure of the core forced out one of the faces. The core was then built up of dry material which was packed by rolling.

37. Spillways.—In every case where a dam is built there must be some provision for permitting excess water to run away so that a specified height of water in the reservoir shall not be exceeded. The device provided for this purpose is called a *spillway*. It may be located at a different place from the dam so that no water ever overtops the latter, as in Fig. 38. Again, the spillway may occupy a portion or all of the crest of the dam, as in Fig. 39. In the latter case the height of the dam must naturally be less than the maximum allowable height of the water.

Spillways must be carefully designed so as to dissipate the kinetic energy of the water without permitting the stream bed to be scoured out or other damage done which might undermine the dam itself or other structures.

38. Flashboards.—In storing water by means of a dam it is desirable to keep the water level as high as possible without

flooding any lands upstream. If, therefore, the crest of the spillway were located at the elevation allowable under normal conditions it would be excessively high in times of flood. In order to

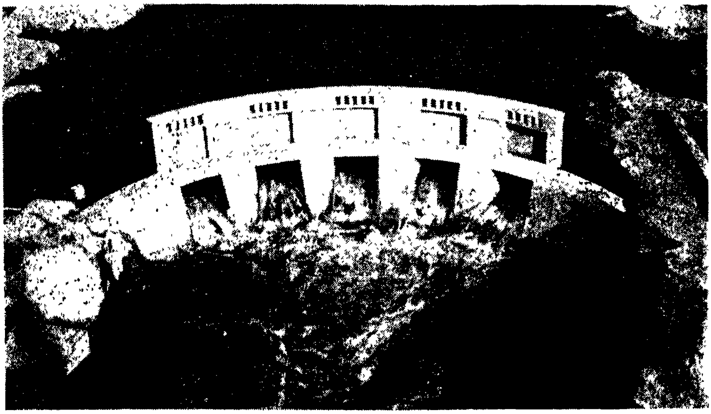

FIG. 39.—Model of Parker dam on Colorado River with a flow over the spillway proportional to 229,000 cu. ft. per sec. (*Courtesy of Metropolitan water District of Southern California.*)

overcome this difficulty movable devices are employed called *flashboards*, *movable crests*, and various other names (Fig. 40). These are all schemes for increasing the height of the dam by equipment that can be removed when necessary. In some cases

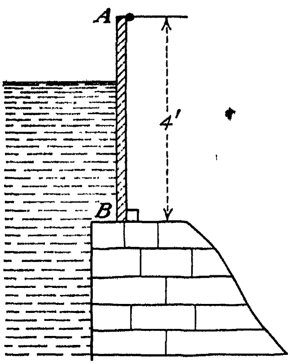

FIG. 40.—Flashboard.

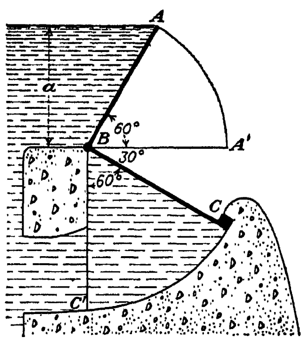

FIG. 41.—Automatic dam crest.

they work automatically, being either washed away when the water reaches a certain stage or caused to drop to a horizontal position. Other types require removal by hand in such emergencies. After the flood is past and the drier season comes

on they may be replaced again. Some of these are entirely automatic in their action, as in the case of the Stickney automatic crest outlined in Fig. 41. We have here two planes AB and BC rigidly connected and rotating about B. The water pressure on AB together with the weight of the shutters and the additional weight added at C tend to rotate the device in one direction, but that is opposed by the pressure of the water on BC. By a suitable adjustment of area and weights it is possible to keep this crest in the position shown until the water reaches the level of A. Then the pressure on AB may be sufficient to cause it to drop to the position $A'BC'$. Hence, the crest of the dam will then be reduced to the height of B, and the flood water will pour over the shutter BA' and hold it down. But when the excess waters have passed and the water level drops to B or thereabouts, the pressure on BC', no longer opposed by that on BA', will raise the crest to the initial position.

39. PROBLEMS

60. Assume the weight of the concrete used for the dam in Fig. 32 to be 150 lb. per cu. ft., the cross-sectional area of the dam to be 206,000 sq. ft., its center of gravity to be 263 ft. from B, the cross-sectional area of the water ABA' to be 54,400 sq. ft., and its center of gravity to be 40 ft. from B, both distances being measured horizontally, and neglect uplift from leakage. Find the ratio of the horizontal water pressure to the total vertical force, the factor of safety against overturning, and the location of the point where the resultant pressure cuts the base. *Ans.* 0.47, 3.65, 299 ft. from toe.

61. For the dam shown in Fig. 35, assume the weight of concrete to be 150 lb. per cu. ft., the cross-sectional area of the dam 45,195 sq. ft., the face at the top to be 16 ft. from a vertical through the heel, the water pressure on the base to vary uniformly from 329 ft. at the heel to 50 per cent of that value at the drainage wells 24 ft. from the heel and then uniformly to zero at the toe, the value of K to be 1, and the height of water in the reservoir to be 329 ft. Find the ratio of the horizontal water pressure to the total vertical force. *Ans.* 0.642.

62. At a height 100 ft. above the base, the cross-sectional area of the dam in Fig. 35 above that level is 21,773 sq. ft., the distance from the face to the drainage wells is 19 ft., and all other conditions are as assumed in Prob. 61. Find the ratio of the horizontal water pressure to the total vertical force. *Ans.* 0.648.

63. The center of gravity of the area representing uplift pressure on the base in Fig. 35 is 190.5 ft. from the toe. With the other data given in Prob. 61 find the point where the resultant pressure cuts the base. *Ans.* 109 ft. from toe.

64. In Fig. 34 the pressure at any point a distance y from the center is given by $p = \frac{R_v}{L} + \frac{R_v 12by}{L^3}$, where b is the distance of R_v from the center and both b and y are plus if measured to the right of center and minus if measured to the left. Find vertical stresses at B and O for the dam shown in Fig. 35. *Ans.* 48.0 lb. per sq. in., 215.6 lb. per sq. in.

65. In Prob. 61 assume that drainage wells had not existed and that the pressure varied uniformly from heel to toe but that $K = \frac{2}{3}$. Find where the resultant pressure cuts the base. *Ans.* 108 ft. from toe.

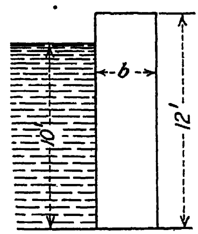

FIG. 42.

66. What value of b in Fig. 42 is necessary to keep the masonry wall from sliding? Masonry weighs 150 lb. per cubic foot, and the coefficient of friction equals 0.4. Will it also be safe from overturning? If it has a factor of safety against sliding of 2, where will the resultant of the water pressure and its weight cut the base? Neglect uplift.

Ans. 4.34 ft., 3.67 ft. from toe.

67. A dam has a trapezoidal cross section with the water face vertical. The base is 20 ft. wide, the top 2 ft. wide, and the dam is 20 ft. high. The depth of water is 18 ft. The dam weighs 150 lb. per cu. ft. Find where the resultant pressure cuts the base. Find the stress on the base at the heel and at the toe.

Ans. 11.44 ft. from toe, 16.4 lb. per sq. in., 6.5 lb. per sq. in.

68. Assume the weight of the dam in Fig. 43 to be 150 lb. per cubic foot, that there is no seepage of water under its base, and that the coefficient of friction between the dam and the material upon which it rests is 0.6. For 1-ft. length compute (*a*) horizontal component of water pressure; (*b*) vertical component of water pressure; (*c*) weight of dam. (*d*) Is it safe against sliding? (*e*) Is it safe against overturning? (*f*) Where does the resultant of the water pressure and the weight of the dam cut the base?

Ans. (*f*) 15.55 ft. from toe.

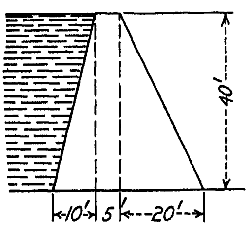

FIG. 43.

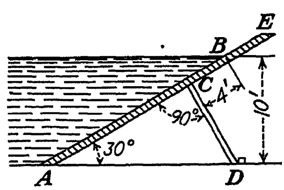

FIG. 44.

69. In the framed dam shown in Fig. 44, the struts CD are placed 5 ft. apart along the dam (perpendicular to the plane of the figure). What will be the load on each strut? What will be the value of the reaction at A? If the length BE is 4 ft. and the depth of the water flowing over the crest at E is 3 ft., what will be the load on the strut?

Ans. 13,030 lb., 5 × 3,630 lb., 39,400 lb.

70. Solve Prob. 66 assuming that water-pressure diagram on base is a triangle and that $K = 0.6$.

71. Solve Prob. 67 assuming that uplift diagram is triangular and that $K = 0.5$.

72. Solve Prob. 67 for an empty reservoir.

Ans. 13.27 ft. from toe, 22.7 lb. per sq. in., 0.21 lb. per sq. in.

73. In Fig. 40 the flashboard AB rests against a solid block at B, but there is a pin at either end at A which is breakable. If the length of a section of flashboard is 6 ft., what must be the shearing strength of the pins if they give way when the water level reaches A?

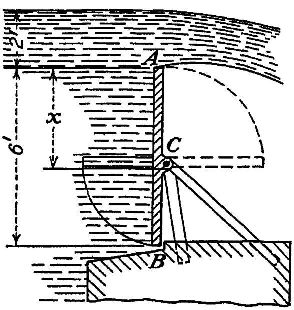

FIG. 45.

74. In Fig. 41 let $BC = 5.0$ ft. and assume that the weight of the plates AB and BC together with the added weight at C is equivalent to a concentrated weight at C of 250 lb. per foot length of crest. (*a*) What will be the force per foot of length exerted against the stop at C when $a = 4.0$ ft.? (*b*) What will be the value of a above which the crest will tip over?

Ans. (*a*) 7900 lb. vertical or 9130 lb. normal to BC. (*b*) 7.65 ft.

75. A rectangular flashboard AB in Fig. 45 is pivoted at C. It is required that the flashboard tip over when the height of water exceeds 2 ft. above A. Where shall the pivot C be placed? What is the distance between the center of gravity of the flashboard and the center of pressure? What is the total pressure per foot of length? *Ans.* $AC = 2.4$ ft.; $P = 1{,}872$ lb.

76. The flashboard of the preceding problem is pivoted at C where $x = 3.5$ ft. What minimum force exerted horizontally at A is necessary to tip it over? What height of water (instead of 2 ft.) would reduce this force to zero?

CHAPTER V

KINEMATICS OF FLUID FLOW

40. Ideal Conditions.—In the ideal case of a frictionless or nonviscous fluid flowing in a channel, such as Fig. 46, all particles move in parallel straight lines with equal velocities. The magnitude of the velocity is shown as OB, and the velocity profile is ABC.

In the actual case of a real fluid the foregoing conditions are modified owing to the effect of viscosity, as will be shown later. Thus it will be found that different particles move with different velocities and that in most cases the lines are not straight. In fact, an individual particle may follow a very irregular and erratic path, even in what appears to the eye to be a very smoothly flowing stream, and its velocity may vary in both magnitude and direction. Furthermore, no two particles have identical or similar motions. A rigid mathematical treatment of such a case can be seen to be very difficult. It will be discussed later, but for the present we shall deal with ideal fluids or with fundamental principles which can be applied to ideal and actual fluids alike.

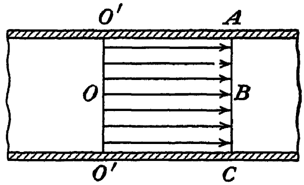

FIG. 46.

In many cases in dealing with fluids of relatively small viscosity such as water or air (see Fig. 2) the results obtained by considering the fluid as frictionless are often of practical value. If properly interpreted or slightly modified, they may be made to yield quite accurate values. In view of this fact there is justification for taking advantage of the great simplification that is afforded by either considering the fluid as frictionless or else, even when friction is considered, of idealizing the flow.

41. Rate of Discharge and Mean Velocity.—The quantity of fluid flowing across any section per unit time is called the *rate of discharge.* It may be expressed in any suitable units such as

pounds per hour, cubic feet per minute, gallons per day, etc. In the foot-pound-second system of units, such as is employed in this book, it would naturally be in pounds per second or cubic feet per second. The latter is often called "second feet" for brevity and abbreviated as "sec.-ft."[1]

In the ideal case shown in Fig. 46 the rate of discharge is obtained by multiplying the velocity OB by the cross-sectional area of the stream normal to OB. If, however, the velocity is not uniform over the section, as is the case in reality, then

$$q = \int_O^A V' \, dA = AV. \tag{15}$$

where V' is the velocity through an infinitesimal area dA, while V is the mean or average velocity over the entire area A. If V' is known as a continuous function, the foregoing may be integrated by calculus or by other methods. If, however, average values of V are known for different *finite* areas into which the total area may be divided, then

$$q = A_1V_1 + A_2V_2 + \cdots + A_nV_n = \sum V'\Delta A = AV. \tag{16}$$

If the rate of discharge has been determined by some method, the *mean* or *average velocity* may be found by

$$V = \frac{W}{wA} = \frac{q}{A}. \tag{17}$$

The value OB in Fig. 46 is equivalent to the average velocity in the actual case.

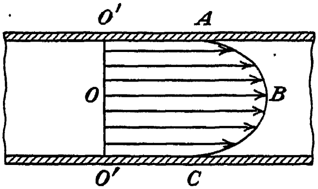

FIG. 47.

EXAMPLES

77. It is sometimes assumed, though there is no theoretical reason for it, that the velocity profile in a pipe of circular cross section is the semi-ellipse ABC shown in Fig. 47 and that the center velocity OB is twice the velocity near the wall. The rate of discharge $V' \, dA$ is seen to be represented by the volume of the solid $O'ABCO'$. Therefore, the mean velocity is represented by the length of a cylinder of the same volume and base. The volume of an ellipsoid is two-

[1] In India the term "cusec" has gained wide acceptance for this quantity. In the western part of the United States the "miner's inch" is frequently employed as a unit of measure. Its value varies from 1/40 to 1/50 cu. ft. per sec. according to legal statute or custom in the region.

thirds that of the circumscribing cylinder. Prove that for this assumption the ratio of the mean velocity to the maximum velocity is 0.833.

78. It is sometimes assumed, though it is not strictly true, that the velocity profile for a circular pipe is a parabola from the center to a point near the wall where the velocity is about 0.7 times the center velocity. The volume of a paraboloid is one-half that of the circumscribing cylinder. If the 0.7 center velocity value is close enough to the wall for the flow between that point and the wall to be neglected, prove that the ratio of the mean to the maximum velocity is 0.85.

79. The flow of a river may be obtained by measuring the velocity at different points in a cross section. Suppose that a certain stream is divided into six different areas of 10, 8, 6, 5, 7, and 9 sq. ft. and that the velocity in each area is found to be 2, 4, 5, 6, 5, and 3 ft. per sec., respectively. Find rate of discharge and mean velocity.

Ans. 174 cu. ft. per sec., 3.86 ft. per sec.

80. The velocities in a circular conduit 20 in. in diameter were measured at the following radii: 0, 3.162, 4.4472, 5.477, 6.325, 7.071, 7.746, 8.367, 8.944, 9.487, and 10 in. and were found to be 5.00, 4.73, 4.50, 4.28, 4.10, 3.88, 3.65, 3.40, 3.00, 2.55, and 1.60 ft. per sec., respectively. Find rate of discharge and mean velocity. This may be solved graphically. Since $q = \int V'dA = \int V'd(\pi r^2)$, it follows that q is π times the area under a curve of V' plotted against simultaneous values of r^2, such as is shown in Fig. 48, where r_0 is the radius to the wall. (See footnote in Art. 21.) In cases where the difference between the maximum and minimum velocities is small compared to the actual values, greater accuracy may be obtained by plotting and measuring only the area above the line xy in Fig. 48 and adding to the result the value of the rectangle below xy. In this particular problem, since the radii were so chosen as to secure equal steps in the values of r^2, the answer may be obtained without actually plotting the curve. Since the ordinates are all spaced at *equal* intervals of 10 sq. in. ($= {}^{10}\!/_{144}$ sq. ft.), Simpson's rule or some other method may be used.

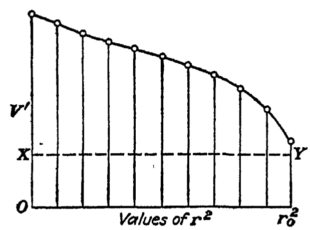

FIG. 48.

Ans. 8.18 cu. ft. per sec., 3.75 ft. per sec.

81. Velocity measurements in a circular jet of water 8 in. in diameter at radii of 0, 1, 2, 3, 3.25, 3.5, 3.75, and 3.9 in. gave velocities of 300, 298, 294, 284, 281, 276, 266, and 256 ft. per sec., respectively. By extrapolation after plotting, it can be estimated that the velocity at the edge of the jet (r_0 in Fig. 48) was about 248. Find rate of discharge and mean velocity. This can be solved by plotting and measuring the area above xy in Fig. 48; or other values may be scaled from the curve, and Simpson's rule used.

Ans. 98.8 cu. ft. per sec., 283 ft. per sec. approx.

42. Steady Flow.—By steady flow is meant that at any point in a stream all conditions remain constant with respect to *time.*

This does not mean that conditions at any one point are necessarily like those at some *other* point in *space.*

However, in most cases of flow there exist continual fluctuations in both velocity and pressure at every point, due to the irregular motions of individual particles. This is the reason that

FIG. 49.—The Los Angeles Aqueduct.

manometers or pressure gages attached to a pipe, in which fluid is flowing, usually show pulsations. But over a period of time the values will fluctuate equally on both sides of a *constant average* value. Another example may be seen in Fig. 49, the dark band on either side of the channel being where the water has wet the concrete by wave action. Yet over a period of time the water surface in the canal will have neither risen nor fallen. Such cases, where the average values of velocities, pressures, and other

quantities remain constant over a period of time, may be known as *mean steady flow*. However, the term *steady flow* without any qualification is generally understood to include this latter type and is not restricted to the exact definition given in the first paragraph.

By contrast with the preceding, *unsteady flow* is the case where the average values vary continuously as a function of time. One example of this may be seen in Fig. 50, where (*a*) denotes the surface of a stream of water that has recently been admitted to the stream bed *xy* of an open canal by the opening of a gate. After a time the water surface will be at (*b*) and later at (*c*) and finally will reach (*d*) where equilibrium is attained and no further change occurs. While the water is rising, the flow is unsteady. When it reaches equilibrium at (*d*) it becomes steady, in the broader sense of the word, although it is true that there will still be minor fluctuations.

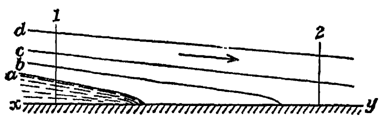

FIG. 50.

As another example of unsteady flow, consider the result of suddenly closing a valve at the discharge end of a long pipe line that is flowing full of water. The velocity in the pipe line is quickly reduced to zero. In the meantime there are rapid fluctuations in pressure, which are initially quite large in magnitude and give rise to the phenomenon known as *water hammer*.

43. Equation of Continuity.—In Fig. 50 it is seen that when the water surface is at (*a*) there is a rate of discharge q_1 past section (1), but q_2 is zero. And when the water surface is at (*c*), q_1 must be greater than q_2 in order to supply the increasing quantity of fluid which is accumulating between (1) and (2). But when equilibrium is reached, the volume between (1) and (2) becomes constant. Even if the fluid is compressible, the total weight between (1) and (2) must then remain constant. Thus the rate at which fluid flows in must equal the rate at which it flows out. That is for steady flow

$$W = w_1A_1V_1 = w_2A_2V_2 = \cdots = wAV = \text{constant.} \qquad (18)$$

If the variation in specific weight is negligible, this reduces to

$$q = A_1V_1 = A_2V_2 = \cdots = AV = \text{constant.} \qquad (19)$$

In order to use Eq. (18) it is necessary to know the way in which the specific weight varies with pressure and temperature. But for an incompressible fluid, Eq. (19) shows that the velocity varies inversely as the cross-sectional area of the stream. If the latter is known, the variation in velocity along the length of the stream is determined (see Fig. 54). The equation of continuity applies equally both to an ideal fluid and to an actual fluid, provided the mean velocity is employed in the case of the latter.[1]

[1] In theoretical derivations it is often necessary to use the continuity principle in the form of a differential equation. It is obtained as follows: Figure 51 shows three coordinate axes x, y, z mutually perpendicular to each other. Let the velocity components in those three directions be u, v, w, respectively. (Note that this present use of w is not in accord with the notation generally used throughout the book.) For steady flow the parallelopiped with sides dx, dy, dz must contain a constant mass of substance. In the x direction the quantity of fluid entering this elementary volume is $\rho u\, dy\, dz$, while the quantity flowing out through the opposite face is $\left[\rho u + \frac{\partial(\rho u)}{\partial x} dx\right] dy\, dz$. Thus the excess (or deficiency) of fluid flowing out in the x direction is $\frac{\partial(\rho u)}{\partial x} dx\, dy\, dz$. Similar expressions may be obtained for the y and z directions. Since the flow is steady, ρ does not vary with time, but it may vary with distance. As the mass in the elementary volume is constant, the sum of the excesses in the three directions must be zero. Since $dx\, dy\, dz$, which is common to all three, is not zero; and since

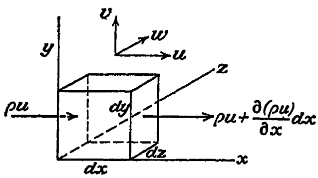

FIG. 51.

$$\frac{\partial(\rho u)}{\partial x} = \frac{u\,\partial\rho}{\partial x} + \frac{\rho\,\partial u}{\partial x},$$

it follows that for steady flow the general equation of continuity is

$$u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0. \qquad (20)$$

In case the flow is not steady, the sum of the three excesses should be equated to $-(\partial\rho/\partial t)\, dx\, dy\, dz$, which results in $+\partial\rho/\partial t$ appearing in the left-hand side of Eq. (20). In the case of an incompressible fluid, whether the flow is steady or unsteady, the continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad (21)$$

The differential equations for continuity are seen to be more general in that they apply to three-dimensional flow. They are of use in advanced work in hydrodynamics.

EXAMPLES

82. The canal shown in Fig. 178 is 14.5 ft. wide and 4.2 ft. deep. The rate of discharge is 313 cu ft. per sec. Later this water flows through a penstock (Fig. 281) which is 52 in. in diameter and then is discharged through four nozzles the jets from which are approximately 7 in. in diameter. What are the velocities? *Ans.* 5.15, 21.2, and 292 ft. per sec.

83. Steam flows through a nozzle at the rate of 2 lb. per sec. At sections where the velocities are 812, 1,333, 1,493, 1,920, 2,490, 3,260 ft. per sec., the specific weights are 0.378, 0.300, 0.270, 0.195, 0.119, 0.031 lb. per cu. ft.. respectively. Find the areas.

Ans. First section, 0.00653 sq. ft.; last section, 0.01975 sq. ft.

44. Path Lines and Streamlines.—A *path line* is the trace made by a *single* particle over a *period* of time. If a camera were to take a long time exposure of a flow in which there were a few particles so colored that they would register on the negative, the picture would show the course followed by each one, which would be its path line. Thus the path line shows the direction of the velocity of the *same* particle at *successive* instants of time.

A *streamline* shows the directions of the velocities of a *number* of particles at the *same* instant of time. If a camera were to take a very short time exposure of a flow in which there were a large number of particles, each one would trace a very short path which would indicate its velocity during that brief interval. A series of curves drawn tangent to these velocity vectors are the streamlines. In general, the streamline diagram will vary from time to time, except in steady flow of a frictionless fluid. Path lines and streamlines are identical for a perfect fluid in steady flow. This is because particles always move *along* streamlines, since these lines show the direction of motion of every particle; and in steady flow the streamline diagram of an ideal fluid does not change with time.

In Fig. 52 a single particle over a long enough period of time would traverse one of the curves there shown. But if a series of particles along one of these curves were observed for a brief time interval, each one would move a short distance along the same curve, the distance moved being proportional to its velocity. The diagram in Fig. 52 was drawn to scale for an ideal fluid, but it is quite accurate even for an actual fluid such as water or air, except for a very thin layer next to the surface of the body. That is, it is quite accurate for the front end of such a body. A similar

diagram constructed for the rear of a "streamline" body would not be quite so reliable, especially close to the "axis," and would be quite inaccurate in the case of flow in the rear of a bluff-shaped body, owing to the eddies that arise here in the case of a real fluid.

It should be observed that we are really concerned with the *relative* velocity between the fluid and the body. For the ideal case the effect is the same whether the body is at rest and the fluid flowing past it or the fluid is at rest and the body moving through it. For the two cases to be equivalent, the undisturbed velocity

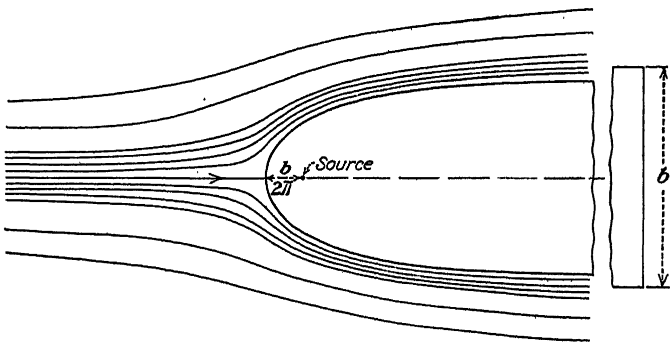

FIG. 52.—Two-dimensional flow of a frictionless fluid past a solid whose surface is perpendicular to the plane of the paper. Streamlines or path lines for steady flow. Observer at rest with respect to the body.

of the fluid at an infinite distance must be the same as the velocity of the body through the stationary fluid. Practically, the effect of the disturbance is inappreciable at a moderate distance. For a real fluid it is probable that the resistance is slightly greater when the body is at rest than it is when the fluid is at rest.

In describing a flow there are two *frames of reference* that may be used. Thus in Fig. 52, which might be a bridge pier, for example, the observer (or the camera) is stationary. But if the solid were a ship moving through still water, the observer would be moving with the ship. In either case he is at rest with respect to the *solid*.

On the other hand, if the observer were floating with the current past the pier in the one case or were in a fixed position while the ship went past in the other, the streamlines and path lines would appear to him as in Fig. 53. In both these cases the observer (or the camera) is at rest with respect to the undisturbed *fluid*.

It is seen that merely changing the frame of reference transforms a steady into an unsteady flow. In one case the velocities and paths are relative to the object, and in the other they are relative to the undisturbed fluid. Velocities and paths relative to a base such as the earth are sometimes called *absolute velocities* and *absolute paths.*

It is obvious that particles in front of any round-nosed object moving through a fluid will be pushed forward and to one side. If the body is given a special shape, as is the case in Fig. 53, all

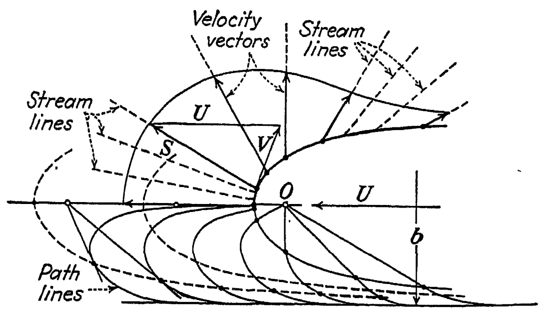

FIG. 53.—Two-dimensional flow of a frictionless fluid showing streamlines and path lines for unsteady flow. Observer at rest with respect to the undisturbed fluid.

the streamlines will radiate from a single point. If a large number of particles in Fig. 53 were photographed for a very brief interval of time, they would all move minute distances, as indicated by the short dashes, but all along radial lines. For a slightly longer time interval the dashes would be seen to be slightly curved, but the tangents to them would all be radial. The particles at the surface of the body would have velocities indicated by the vectors shown. The velocities of particles at other positions on the same streamlines would be different in magnitude. After a brief time interval the body would have moved to the left a short distance and carried this streamline diagram along with it. Although the diagram always has the same appearance in itself, the fact that it moves means that the positions of the streamlines, as seen from this viewpoint, are constantly changing with respect to time. Thus the flow is unsteady.

The lower half of Fig. 53, which is plotted to scale, shows the path lines traced by several particles, all of which start out from the surface of the body at the instant that it is in the position shown by the full line. When the body reaches any other position, as indicated by a dotted outline, the particles will still be on the surface of the body but in new locations on that surface. By hypothesis a given particle must always remain in the same streamline in Fig. 52 but not in Fig. 53. Since the surface of the body is a streamline, a particle once on the surface always stays on the surface. Theoretically, the particle at the exact *nose* always remains there, and thus its path is the straight line in the direction of motion, but all other particles flow along the surface of the body and are given a motion of translation at the same time which results in the paths shown.

If the maximum width attained by the body is b, then the path lines must terminate at a distance $b/2$ from the axis of symmetry. In the case of the body shown, this dimension is not reached for an infinite distance, and thus an infinite time is required for the particles to reach the ends of their path lines. But even in the case of a long body of finite length, where the uniform width is reached at a considerable distance, the time required for the particles to traverse the latter portions of their paths will be much greater than for the initial portions.

Since particles flow along streamlines in steady flow and never across them, it follows that streamlines for steady flow are similar to solid boundary walls. Thus in Fig. 52 a constant rate of discharge flows along the *stream tube* between any two streamlines. Therefore the equation of continuity applies to it in the same manner that it does to an entire stream between walls. Hence, for an incompressible fluid we can see that where streamlines are far apart the velocity is low and where they are close together the velocity is high. This is also shown in Fig. 54, where the mean velocity is really determined by the cross-sectional area between solid walls. But in Fig. 52 there is no other boundary wall to confine the fluid. Hence the streamline diagram around the body is the only means of visualizing the variation of the velocity in such a case. Therefore streamline diagrams enable us to use the continuity equation where solid boundary walls enclosing the fluid are lacking and thus serve to give not only a

picture of the directions of the velocities in an entire field but also their magnitudes.

Since streamline diagrams can often be drawn solely from geometrical principles, they provide a valuable means of analysis of flow problems. The plotting of relative and absolute path lines is often desirable in studies of hydraulic turbine runners and centrifugal pump impellers and also in other fields.

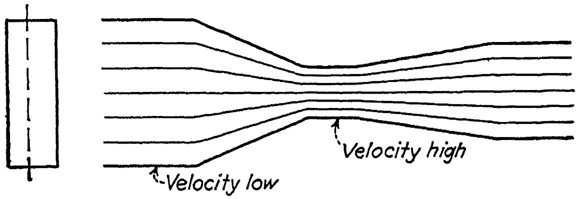

FIG. 54.—Two-dimensional flow of an ideal fluid.

45. Two- and Three-dimensional Flow.—If the flow is such that all streamlines are plane curves and are identical in a series of parallel planes, the flow is said to be two-dimensional. If the coordinates of Fig. 51 are so taken that the z axis is perpendicular to the planes, then the velocity component w is zero, and we are concerned only with the components u and v. In Fig. 54 the channel must have a constant dimension perpendicular to the

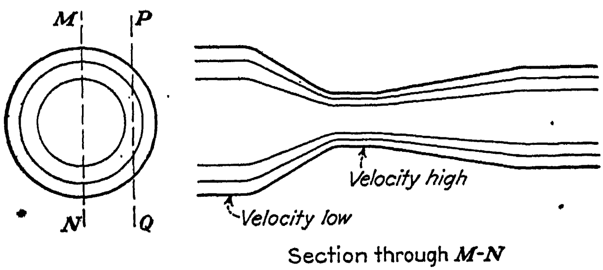

FIG. 55.—Three-dimensional flow of an ideal fluid.

plane of the figure. Thus every cross section normal to the flow must be a rectangle with one side of a constant value. If the total rate of discharge is to be divided by the streamlines into equal parts, then the streamlines must be equally spaced, provided that the velocity is uniform over the cross section.

Three-dimensional flow is illustrated in Fig. 55, which differs from the preceding in that the cross sections are circular. The

longitudinal section is taken through the diameter MN, and every streamline in such a section is a plane curve with no velocity component perpendicular to the plane of the paper. But if a section were cut by any other parallel plane PQ, the streamlines would all be different from those shown. It is obvious that the boundary walls for the straight portions would be closer together and that for the conical portions the boundaries would be parabolas. While the velocity component w is still zero for the straight portions, it is not zero for such a section in the conical portions. If the total flow is divided by stream surfaces into equal parts, the streamlines, which are really traces of these surfaces, are not equally spaced (*cf.* Figs. 52 and 56).

Two-dimensional flow offers the advantage that it is usually easier to draw diagrams describing the flow, but the same principles apply to three-dimensional flow.

EXAMPLES

84. It can be proved by the theory of potential functions, though space in this book does not permit, that the equation for the streamlines of a two-dimensional flow of a frictionless incompressible fluid against a flat plate normal to the initial velocity is xy = constant, where the flow is taken as symmetrical about the xz plane (see Fig. 51). A different streamline can be plotted for each value of the constant. Using a scale 1 in. = 6 units of distance, plot streamlines for values of the constant of 16, 64, and 128.

85. For the case in the preceding problem, it can be shown also that the velocity components at any point, whose coordinates are x and y, are $u = ax$ and $v = -ay$, where a is a constant. Thus the actual velocity is

$$V = a\sqrt{x^2 + y^2} = ar,$$

where r is the radius to the point from the origin. Let $a = \frac{1}{3}$; then if 1 in. = 6 ft. for the streamlines, 1 in. = 2 ft. per sec. for the velocity scale. Draw curves of equal velocity for values of 2, 4, 6, 8, 10 ft. per sec. How does the velocity vary along the surface of the plate?

86. For three-dimensional flow with the x axis as the axis of symmetry of a circular stream the streamline equations are xy^2 = constant, and the velocity is given by $V^2 = a^2(4x^2 + y^2)$. Plot streamlines and lines of constant velocity. How does the velocity vary along the plate?

46. Sources and Sinks.—In studying the flow around streamline bodies, such as those shown in Figs. 52, 53, and 56, it is helpful to use what are known as *sources* and *sinks*. A source is a point from which an imaginary flow issues along straight radial streamlines. A sink is a point that receives such a flow. Since an infinite velocity would be required for a finite rate of discharge

to pass through a mathematical point, no such real flow is possible, but it may be approximated in certain cases. Thus a small circular hole in a large flat plate behaves as a sink for a real flow which approaches it from all directions along radial lines, except in the immediate neighborhood of the hole. In Fig. 53 the actual streamlines appear as radial lines, the effect being the same as if the body did not even exist but were replaced by a source flow at O.

In two-dimensional flow, such as in Fig. 53, the source is really a line perpendicular to the plane of the paper at the point O. The flow radiates from this line and passes through the surfaces of cylinders concentric about it. If the height of the cylinders perpendicular to the plane of the paper is y and the source velocity is S through the surface of any cylinder of radius r, then $q_0 = 2\pi r y S =$ constant. Thus $rS =$ constant, or S varies as $1/r$. If the flow is three-dimensional, the source is the center of concentric spherical surfaces. For this case $q_0 = 4\pi r^2 S =$ constant, and thus S varies as $1/r^2$.

If a source flow radiating in all directions from a point has superimposed upon it a rectilinear flow of a uniform velocity U, the resultant velocity at any point in the field can be found by combining vectorially U and S, where the latter is determined by its direction from and distance from the source. The resultant streamlines of the originally rectilinear flow are bent to one side, as shown in Figs. 52 and 56, while the streamlines from the source start out in all directions from the source but are bent around and flow away to the right as shown in Fig. 57. Any streamline may be taken as the surface of a solid body, but the one for which the rectilinear flow and the source flow are equal is the practical one.

The total rate of discharge from the source is called its *strength*. Since the entire source flow must remain inside the streamline which is the surface of the body, its value must be $q_0 = byU$ for two-dimensional and $q_0 = \pi d^2 U/4$ for three-dimensional flow. For bodies of infinite length b and d are the widths and diameters, respectively, at infinity. For bodies of finite length they are the dimensions where the size becomes uniform.

The streamlines here shown are for bodies of infinite length. Practically they apply to bodies so long that the flow at the rear will have a negligible influence at the front. For shorter bodies

the flow in front will be modified. At the rear of any such body there must be a sink which "receives" all the fluid that is "generated" at the source. The source and sink are identical mathematically except that the flow is opposite in sign. If S_1 is the source velocity at a distance r_1, then S_2 is the sink velocity at a distance r_2 from the sink. At the front end of a long body, r_2 will be relatively large and the value of S_2 relatively small. However, it should be combined vectorially with S_1 to get the true source-sink velocity, and this resultant combined in turn with U to obtain the flow picture for such a body.

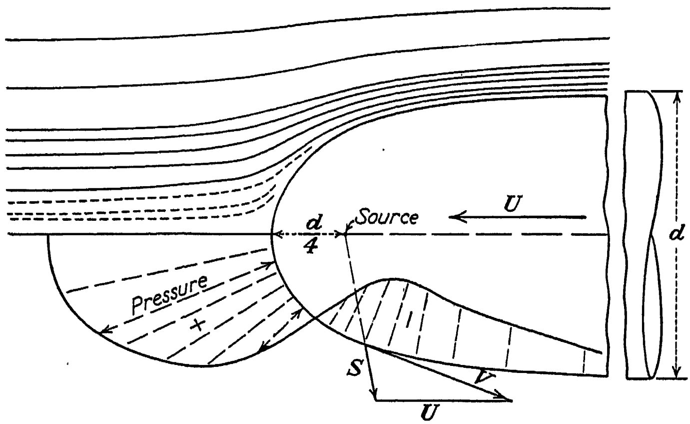

FIG. 56.—Three-dimensional flow of a frictionless fluid past a solid of revolution. The half above the axis shows the streamlines; the lower half shows the pressure variation.

The central streamline in Figs. 52 and 56 indicates that the flow is zero at the very tip of the body. This is called the *stagnation point.* Its distance from the source is found by determining the value of $x = r$ at which $S = U$. Thus $2\pi r y S = b y U$, from which $2\pi x U = bU$, or $x = b/2\pi$. A similar procedure is followed for the three-dimensional flow.

While it is possible to measure the spacing between the streamlines in Figs. 52 and 56 (normal to the mean velocity within the stream tube) and thus obtain the cross-sectional area from which the velocity may be computed, it is difficult to do so accurately without a large and carefully constructed drawing. It is simpler and more accurate to determine the velocity V along any streamline at any point by means of the vector velocity diagram shown in Figs. 53 and 56.

The procedure outlined will enable the plotting of a series of streamlines, such as those shown. One of these streamlines is the surface of a solid body for which all these streamlines apply. However, this is not the only streamline shape that is possible, as one may use a number of sources and sinks, distributed along the axis instead of the single points to which the present discussion is confined, and thus produce other streamline shapes. The reverse problem of constructing streamlines for a given shape of body is much more difficult, since it requires solution by trial and approximation.

Although this theoretical treatment is for perfect fluids, observations of velocity and pressure fields for the flow of both water and air around streamline bodies show close agreement with the theory except near the extreme tail and in the small *wake* following. The method may be applied to such cases as the flow past the vane of a turbine runner or centrifugal pump impeller or to the flow past an airplane strut or airplane wing. These may be considered as approximately two-dimensional flow. However, in the case of the wing, for example, such a flow would be obtained only for an infinite length of wing, since the flow around the tip is different from that in the middle. The flow around an airship body or around a submarine, when submerged, are examples of three-dimensional flow.

47. Plotting Stream and Path Lines.—In the case of the source-sink method of determining the flow picture past a streamline body, the graphical construction will be explained for the infinitely long body, but the procedure will be similar in the case of a short body, the difference being the complication due to the inclusion of a sink velocity vector for the latter.

In Fig. 57 the streamlines for a two-dimensional flow are obtained by drawing curves through the intersections of a series of parallel lines spaced an equal distance apart and a series of radial lines from the source which are laid off at equal angles. The proof for this is: The uniform flow from the left supplies a rate of discharge $q_1 = ayU$, while the source supplies

$$q_2 = \left(\frac{\alpha}{360}\right)q_0 = \left(\frac{\alpha}{360}\right)byU. \tag{22}$$

Since there can be no flow across the boundary wall of the body, these two must be equal, or $q_1 = q_2$. Thus $a = (b/360)\alpha$. The

point on the radial line at angle α whose height above the center line is a is a point on the surface of the body. Since a is directly proportional to α, it follows that $2a$ is associated with 2α; $3a$ with 3α; etc. Now, the surface of the body is a streamline, and between it and any other streamline must flow a constant quantity Δq. Then $q' = q_2 + \Delta q$, where q' is a larger value of q_1 flowing through an area determined by $a + \Delta a$. It is convenient to take Δq equal to q_2, and then Δa becomes equal to a. If all values of the constant quantity are taken as multiples of Δq, then the same parallel lines drawn for the body surface can be used for all the other streamlines.

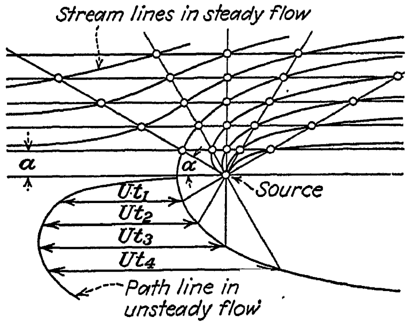

FIG. 57.—Plotting of stream and path lines.

The reasoning is the same for the three-dimensional flow. The flow q_1 is through a cylinder and is $q_1 = \pi a^2 U$. The flow from the source is that through the portion of the surface of a sphere which is cut out by a cone with a vertex angle 2α and is $q_2 = q_0(1 - \cos\alpha)/2 = (1 - \cos\alpha)\pi d^2 U/8$. Equating these two, the surface is found by $a = d\sqrt{(1 - \cos\alpha)/8}$. Other streamlines can be found by

$$a_n^2 = \frac{d^2(1 - \cos\alpha)}{8} + \text{constant} = a^2 + \text{constant},$$

where a is the coordinate for the surface of the body. Since the spacings between the horizontal lines will be variable, there is no special convenience in using any particular values of the constant; but if it is desired to have the stream surfaces enclose equal rates of discharge, the constants may be taken as multiples of the initial q_2, as in the preceding case. However, this is not necessary and was not adhered to altogether for Fig. 56.

To plot path lines for the case as shown in Fig. 53, the procedure is as follows: Measure values of r from the source to different points along the surface and then compute the value of the source velocity at each point. Combine vectorially each source velocity with the uniform velocity U. This is easily done graphically. Measure on the drawing the distances along the surface between these points. Near the nose, where the curvature becomes marked, these distances may be computed as arcs of circles with radii the mean of the two values at either end. The velocity in each portion may then be taken as the mean of the two velocities at each end, and the time computed for a particle to flow along the surface from one point to the next. Beginning with a particle at some point on the surface at or near the nose, add values of these time intervals so as to find the time for this particle to reach successive points on the surface. Also, compute how far each one of these points on the surface will have moved owing to its uniform velocity U during that same period of time and lay off these distances from the initial surface curve. These points determine a stream path. Since all stream paths are similar, others may be drawn from this one by a mere process of translation, most easily done by means of a paper template. If desired, points on all the stream paths that are reached in the same instant of time may be found by drawing tangents to the curves from the position occupied by the source at that instant. This method of plotting path lines is easy and is also accurate.

By these and other geometrical methods, pictures of flow may be obtained. These space-time relationships which have been discussed so far in this chapter constitute the field of kinematics. We shall next take up the consideration of forces and energy, which constitute the subject of kinetics or dynamics.

EXAMPLES

87. Assume a value of $U = 20$ ft. per sec. and a two-dimensional flow past a body for which $b = 36$ ft. Compute the distances from the source to the stagnation point and to the surface of the body at a radius 90 deg. to the axis. What is the value of the source velocity at the latter point?

Ans. 5.73 ft., 9.00 ft., 12.75 ft. per sec.

88. What is the magnitude of the velocity of the fluid along the surface at the 90-deg. point in the preceding problem? What is its direction relative to the axis?

Ans. 23.7 ft. per sec., 32.5 deg.

89. Find the distance and the two velocities called for in the preceding two problems for an angle of 30 deg.

Ans. 6.00 ft., 19.12 ft. per sec., 10.15 ft. per sec.

90. For a three-dimensional flow around a body of revolution, compute radii from source to surface for 0, 30, 60, 90 deg. if $d = 4$ ft.

Ans. 1.00, 1.034, 1.155, 1.414 ft.

91. If $U = 20$ ft. per sec., find source velocities at the points in the preceding problem and also the velocities along the surface.

Ans. For 30 deg., $S = 18.65$ ft. per sec., $V = 10.1$ ft. per sec.

92. Plot boundary of body and set of streamlines for a steady two-dimensional flow for $b = 36$ ft., using a scale of 1 in. = 6 ft.

93. Plot streamlines and path lines for the preceding for the case where the observer is at rest with respect to the undisturbed fluid.

CHAPTER VI

DYNAMICS OF FLUID FLOW

48. Kinetic Energy of Flowing Stream.—If all particles of a stream flow with the same velocity V over an entire cross section, as in the ideal case shown in Fig. 46, the mass that passes the section per second is $m = \rho AV$, and the kinetic energy that this mass possesses, and therefore carries across the section per second, is $mV^2/2$. Since $m = W/g$, this may also be expressed as

$$\text{Kinetic energy per second} = \frac{WV^2}{2g}, \tag{23}$$

from which it follows that

$$\text{Kinetic energy per pound of fluid} = \frac{V^2}{2g}. \tag{24}$$

But in the actual case of a real fluid the velocities of different particles will usually not be the same. Consider the case where the velocities are all in straight axial streamlines but vary in magnitude over the cross section, such as is shown in Fig. 47. If V' is the velocity at any point, the kinetic energy per second transmitted across an elementary area is $dW\ V'^2/2g$. Since $dW = wV'\,dA$, therefore for the entire cross section,

$$\text{Kinetic energy per second} = \frac{w}{2g}\int_0^A V'^3\,dA. \tag{25}$$

For any case where the velocity profile is known the kinetic energy may be found by the aid of this equation, but for illustrative purposes it may be convenient to state this as follows:

$$\text{Kinetic energy per second} = \frac{1}{2g}\int_0^W V'^2\,dW = W\frac{\overline{V^2}}{2g}, \tag{26}$$

where $\overline{V^2}$ is the average with respect to W of V'^2. It can be shown that the average of the squares of the velocities is always

greater than the square of the average velocity and therefore that

$$\text{Kinetic energy per pound of fluid} = \frac{V^2}{2g} + N^2, \tag{27}$$

where N^2 is added to take account of the fact that the velocity profile is not a straight line.

In case the entire velocity profile in a circular pipe is a parabola extending to zero velocity at the wall, which is a case that may actually exist, the kinetic energy per average pound flowing across the section is $2V^2/2g$, or twice the value given by Eq. (24). On the other hand, if the velocity profile were a semi-ellipse with the velocity near the wall one-half that in the center, the kinetic energy per average pound would be only $1.06V^2/2g$, while for a jet of water from a good nozzle the value is about $1.01V^2/2g$.

Also, in many common cases the flow is not strictly steady but is really mean steady flow. That is, there are continual fluctuations in velocity at a point both in direction and in magnitude. In this case the mean velocity V is not only the average with respect to the area but is also the average at that area with respect to time. If the rate of discharge is to be the same as in the previous cases, the value of V must remain the same, but it may be considered to have superimposed upon it variable velocity components both axially and in all other directions. These components add to the kinetic energy of the stream an amount that will be represented by B^2. Then the total kinetic energy of turbulent flow is greater than given by Eq. (27), and

$$\text{Kinetic energy per pound} = \frac{V^2}{2g} + N^2 + B^2. \tag{28}$$

This may also be represented by $\alpha \dfrac{V^2}{2g}$ where α is a factor greater than unity.

EXAMPLE

94. Assume that in an open canal the velocity from surface to bottom varies as a straight line and that the surface velocity is twice the bottom velocity but that the velocity in any horizontal line is uniform. Find the value of N^2. *Ans.* $0.11V^2/2g$.

49. General Energy Equation for Steady Flow of Any Fluid.—In Fig. 58 is shown a stream to which the law of conservation of energy will be applied to the portion between cross sections (1)

and (2). At section (1) the total force on the area is p_1A_1. Assume that in a brief time dt this section is moved a short distance ds_1; then the force p_1A_1 will do an amount of work $p_1A_1\,ds_1$ on the body of fluid between (1) and (2). Since work and energy are equivalent, this means that this amount of energy is transmitted across the section. In a similar manner the amount of work done at (2) is $-p_2A_2\,ds_2$, the minus sign being used because the force and the displacement are in opposite directions. Hence the net amount of work done upon the body of fluid between (1) and (2), due to the pressures at the two ends, is $p_1A_1\,ds_1 - p_2A_2\,ds_2$. This also represents the net delivery of

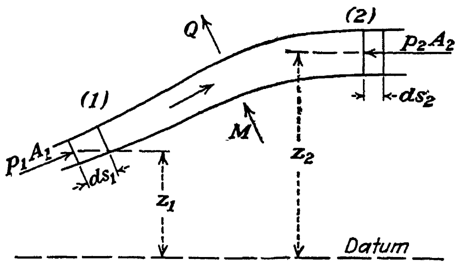

FIG. 58.

energy to the body of fluid under consideration due to the work done by these forces.

During this same period of time the body of fluid between (1) and (2) gains the energy that is brought into it by the fluid entering at (1) and loses that which is carried out by the fluid leaving at (2). Also, it will be assumed that between (1) and (2) thermal energy may be *lost* by the conduction of heat through the channel walls and "radiated" or transferred by some means to the surroundings at the rate of Q British thermal units (B.t.u.) per lb. of fluid flowing. It may also be assumed that between (1) and (2) mechanical energy is *gained* at the rate of M ft.-lb. of work per lb. of fluid flowing, in case there is a machine such as a pump or compressor in operation between these two points.

The energy that a particle actually possesses in itself may be in any one or all of three forms. The first is potential energy due to its elevation z above some arbitrary datum plane. The second is its kinetic energy due to its velocity V, which has been explained in the preceding article. The third is its internal, or

intrinsic, thermal energy I, which is due to its molecular state.[1]

During the time dt the weight of fluid entering at (1) is $w_1A_1\,ds_1$, and the energy carried into the body of fluid by the particles entering is $w_1A_1\,ds_1\left(z_1 + \frac{V_1^2}{2g} + N_1^2 + B_1^2 + 778I_1,\right)$ while that which leaves at (2) is a similar expression. In general, the amount of fluid leaving at (2) may be different from that entering at (1), and also the net result of the flow of energy into and out of the body of fluid may be to change the energy of the volume between (1) and (2). However, we are here restricting ourselves to steady flow, so that the rate at which fluid leaves at (2) is equal to the rate at which it enters at (1), and also there is no change in the energy of the mass of fluid contained between (1) and (2). Thus for steady flow the sum of all of these energy transfers is zero.

Equating the sum of all of these energy terms to zero and factoring out $w_1A_1\,ds_1 = w_2A_2\,ds_2$, the energy equation per unit weight of the fluid is

$$\left(\frac{p_1}{w_1} + z_1 + \frac{V_1^2}{2g}\right) - \left(\frac{p_2}{w_2} + z_2 + \frac{V_2^2}{2g}\right) + 778(I_1 - I_2) + (N_1^2 - N_2^2) + (B_1^2 - B_2^2) - 778Q + M = 0, \quad (29)$$

where 778 is the number of foot-pounds in 1 B.t.u. and is introduced on the assumption that both I and Q are expressed in B.t.u., as is customary. This equation applies to gases, vapors,

[1] For a perfect gas, intrinsic energy is merely a function of its temperature and is $I = c_v(T - T_0)$, where c_v is specific heat at constant volume, T is the temperature, while T_0 is an arbitrary temperature datum. This expression may also be used for actual gases and for liquids, provided the proper numerical value for the specific heat is used. However, it should not be employed for vapors or where a change of state is involved, as in the latter case the internal energy must include the internal latent heat of vaporization.

The distinction between the energy transmitted across a section due to the pressure and motion and that carried across the section as a possession of the particles flowing is similar to the transmission of energy past a point in space by a belt. A belt delivers energy from one pulley to another by virtue of its tension and its motion, but it does not possess this energy itself. But the belt does carry kinetic energy past a point, owing to its mass and its velocity.

or liquids. It applies both to ideal frictionless fluids and to real fluids with friction. The only restriction is that it is for steady flow.

There is no term expressing friction in Eq. (29) because, in this derivation, friction is a purely internal action. Its effect is to transform some of the mechanical energy into thermal energy. That is p_2 or z_2 or V_2, or any combination of them may be reduced by friction, while I_2 or Q or both may be increased a corresponding amount. The total amount of energy leaving the body of fluid under consideration is the same, but it will be differently distributed between mechanical and thermal energy. The value of the friction must be determined by a separate equation, and numerical values for some of the quantities in Eq. (29) altered in accordance therewith, as will be shown later.

The energy equation is the key to the solution of a multitude of problems in hydraulics. It and the continuity equation constitute two important simultaneous equations enabling us to solve for two unknown quantities. For compressible fluids it is necessary to have a third equation, which is the relation between specific weight (or its reciprocal, specific volume) and pressure. This relation for a perfect gas, for example, is p/w^n = constant, where n is given a value which depends upon the process, but this problem takes us into the domain of thermodynamics.

In many cases Eq. (29) is greatly shortened owing to the fact that certain quantities are equal and thus cancel each other or are zero. Thus if two points are at the same elevation, $z_1 - z_2 = 0$. If the conduit is well insulated, or if the temperature of the fluid is practically the same as that of the surrounding medium, Q may be taken as zero. On the other hand, in the case of the flow of water through a boiler tube the value of Q may be very large. If there is no machine between (1) and (2), the term M drops out. On the other hand, if an engine, turbine, pump, or compressor is involved, the work or power may be determined by solving for M, due heed being given to the sign that applies.

The term involving N^2 can be evaluated only by knowing the velocity profile. Fortunately, the quantity is not large in many actual cases, and hence the difference between values of N^2 may be small. If the shapes of the velocity profiles at two cross sections of equal area are similar, then the two values of N^2 cancel. It is even more difficult to determine the value for B^2,

but again in many cases the two values of B^2 cancel or practically so. For the present we shall drop these terms from the equation but shall bring them back in at a later time.

50. Energy Equation for Gases and Vapors.—Where the other quantities are known, Eq. (29) might be used to determine M; but if the problem is to determine something about the flow, the two sections (1) and (2) would be so chosen as to avoid having any machine between them, and thus M would not appear. Therefore, omitting it as well as the N^2 and B^2 terms, the energy equation becomes

$$\left(\frac{p_1}{w_1} + z_1 + \frac{V_1^2}{2g} + 778I_1\right) - \left(\frac{p_2}{w_2} + z_2 + \frac{V_2^2}{2g} + 778I_2\right) = 778Q. \tag{30}$$

For a gas or a vapor the quantity p/w is usually very large because of the small value of w, and therefore z is often omitted as being negligible by comparison. However, the difference between the two values of p/w is sometimes very small, and therefore $z_1 - z_2$ should not be ignored until after it has been found permissible to neglect it.

For a gas, certain relations in thermodynamics, such as $I = c_v(T - T_0)$, $pv/T =$ constant, and $pv^n =$ constant, where v is specific volume or $1/w$, may be combined with the foregoing in various ways and thus produce a variety of forms of Eq. (29).

While this equation can be used for either a gas or a vapor, another form is often convenient. Since in thermodynamics the quantity E, which is variously known as enthalpy, heat content, total heat, or thermal potential, is defined as $778E = 778I + p/w$, the preceding equation may be changed to[1]

$$\left(\frac{z_1}{778} + E_1 + \frac{V_1^2}{50{,}000}\right) - \left(\frac{z_2}{778} + E_2 + \frac{V_2^2}{50{,}000}\right) = Q, \tag{31}$$

since $2g \times 778 = 50{,}000$ approximately. As before, if a machine were involved, the term $\dfrac{-M}{778}$ would appear on the right-hand side of this equation. But if there is no machine, if the difference in elevation is negligible, and if the flow is adiabatic, that is, there

[1] In thermodynamics E is usually represented by the letter h, $1/w$ by specific volume v, and $1/778$ by A, and thus the equation appears in the more familiar form $h = i + Apv$.

is no heat transfer or $Q = 0$, this simplifies to

$$V_2^2 - V_1^2 = 50{,}000(E_1 - E_2). \tag{32}$$

For the vapors commonly used in engineering such as steam, ammonia, and carbon dioxide, values of E may be obtained from vapor tables or charts for any flow process desired.[1]

As has been explained before, friction reduces such terms as p_2 or V_2 or both to values below those which they would have without friction and increases such terms as I_2 or E_2 to values higher than they would be without friction. It also tends to increase the specific volume or decrease its reciprocal, which is the specific weight w_2.

A form of the energy equation involving a friction term may be obtained in the following manner. If the heat of friction is denoted by h_f, the net gain in heat (in foot-pound units) is $h_f - 778Q$. A fundamental energy relation is that the heat added to a body is equal to its increase in intrinsic energy plus the external work done by the body. Writing this as a differential equation, and using v for specific volume,

$$d(h_f - 778Q) = 778dI + p\,dv.$$

From the definition of E, we obtain

$$778dE = 778dI + p\,dv + v\,dp.$$

Inserting this latter value of dI in the former equation,

$$d(h_f - 778Q) = 778dE - v\,dp$$

is obtained. From this we obtain

$$h_f = 778Q + 778(E_2 - E_1) + \int_{p_2}^{p_1} v\,dp. \tag{33}$$

From Eq. (31)

$$778Q + 778(E_2 - E_1) = \left(z_1 + \frac{V_1^2}{2g}\right) - \left(z_2 + \frac{V_2^2}{2g}\right)$$

and inserting this in Eq. (33) we obtain the alternate form

$$h_f = (z_1 - z_2) + \left(\frac{V_1^2 - V_2^2}{2g}\right) + \int_{p_2}^{p_1} v\,dp. \tag{34}$$

[1] For a perfect gas, $E_1 - E_2 = c_p(T_1 - T_2)$, where c_p = specific heat at constant pressure. Thus this form of the equation may be used for gases also, if desired.

In order that $v\,dp$ may be integrated, it is necessary to know the "path," that is, the relation of v (or $1/w$) to p for all values of p between the two limits. Thus for the isothermal flow of a perfect gas, $h_f = 778Q + p_1v_1 \log_e (p_1/p_2)$, which follows at once when pv = constant or $n = 1$. Other types of flow, even if restricted to different possible adiabatic paths, would involve considerable discussion of thermodynamics, which is beyond the scope of this book. Equations (33) and (34) are useful in enabling us to compute friction from experimental data, when all the other values are observed. Since the values of p_2 or V_2 or other quantities in these equations depend upon the value of the friction, it is obvious that the equations cannot be used where the flow conditions are unknown. In such a case the value of the friction must be computed by methods to be shown in a later chapter.

EXAMPLES

95. Dry saturated steam at a pressure of 80 lb. per sq. in. abs. expands in a suitable nozzle to a pressure of 15 lb. per sq. in. abs. For a frictionless adiabatic flow, steam tables give $E_1 = 1{,}182.5$ and $E_2 = 1{,}060.0$ B.t.u. per lb. Assuming the initial velocity to be negligible, find the final velocity. What would it be if the initial velocity were 1,000 ft. per sec.?

Ans. 2,475 ft. per sec., 2,670 ft. per sec.

96. Air at a pressure of 80 lb. per sq. in. abs. and 100°F. expands in a suitable nozzle to a pressure of 15 lb. per sq. in. abs. The flow is assumed to be frictionless and adiabatic, for which $n = c_p/c_v = 0.24/0.1715 = 1.4$. With the aid of this value and the thermodynamics relations, it may be seen that $w_1 = 0.385$, $w_2 = 0.1165$, $T_2 = 347°$ abs. If the initial velocity is negligible, find the final velocity. *Ans.* 1,605 ft. per sec.

97. Air flows isothermally through a long pipe line of uniform diameter. At the section where the pressure is 90 lb. per sq. in. abs. the velocity is 100 ft. per sec. Owing to friction, the pressure at a section farther along the pipe is 15 lb. per sq. in. abs. (*a*) Find the heat transfer per pound of fluid flowing. (*b*) If the diameter of the pipe is 3 in. and the temperature of the air flowing is 80°F., find the total heat transferred per hour in that length of pipe. *Ans.* −7.0 B.t.u. per lb., −55,700 B.t.u. per hr.

98. What is the friction in this length of pipe in the preceding problem per pound of fluid? *Ans.* 46,250 B.t.u. per lb.

99. Throttling a flowing fluid to reduce its pressure may be regarded as converting other energy into kinetic energy through the restricted opening and then having this kinetic energy all dissipated into thermal energy. In this case this represents the friction. Suppose that steam under the conditions in Prob. 95 is throttled to 15 lb. per sq. in. abs. What is the friction loss? *Ans.* 95,200 ft.-lb. per lb.

51. Energy Equation for Liquids.—The energy equation for liquids, and also for gases and vapors where the change in pressure is relatively small, differs from the preceding in that the specific weight is considered as constant. For an incompressible fluid, and omitting the N^2 and B^2 terms, Eq. (29) becomes

$$\left(\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g}\right) - \left(\frac{p_2}{w} + z_2 + \frac{V_2^2}{2g}\right) = 778(I_2 - I_1 + Q) - M. \quad (35)$$

We shall now consider briefly the matter of internal energy. For the present we shall assume that the flow is adiabatic, by which is meant that there is no exchange of heat with the surroundings; that is, $Q = 0$. Adiabatic flow may be frictionless, or there may be friction in which heat is generated by the internal work. Also, external work may change the thermal energy of the fluid, but the process is still adiabatic as long as Q is zero. For a compressible fluid a change in pressure produces a change in internal energy, which may be represented graphically by the area under a curve for a frictionless adiabatic process, as shown in Fig. 59. For an incompressible fluid, this area is zero, since the compression curve is the vertical line ab. Hence the value of $I_2 - I_1$ for this case is zero. Another way of stating this is that compressing a compressible fluid raises its temperature and thus increases its intrinsic energy, while increasing the pressure on an incompressible fluid does not do so, since no work is done in the latter case. Thus even for frictionless flow, I_2 is, in general, different from I_1 for a compressible fluid, while it is not different for an incompressible one.

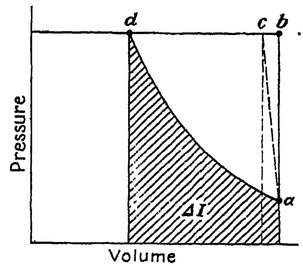

FIG. 59.

But in the case of friction in an incompressible fluid, the internal work converts mechanical energy into thermal energy, thus raising the temperature; and hence, if there is no external heat transfer, I_2 will be greater than I_1. Since $I_2 - I_1$ is multiplied by 778 and $I_2 - I_1 = c_v(T_2 - T_1)$, it is seen that a relatively large friction loss may produce only a very small temperature rise.

For a real liquid, there is a slight change in volume as the pressure varies, and the compression curve *ac* is not strictly vertical, but the area under it is extremely small, and thus the change in intrinsic energy is negligible, except for very large pressure differences. In the case of a gas or a vapor, this area may also be negligible, where the change in pressure is quite small. Hence, for a real liquid, except for extreme cases, and for gases and vapors with small pressure variations, the quantity $I_2 - I_1$ is a measure of the friction, always provided that there is no external heat transfer. This increased thermal energy is not a loss from the system, but it does represent a loss of mechanical energy of the fluid.

On the other hand, if the flow is not adiabatic but the flow of heat Q is at such a rate that the fluid flow is isothermal, then $I_2 = I_1$ despite friction. In this case Q is equal to the friction and does represent an actual energy loss from the system. From the engineer's standpoint it is usually immaterial whether the friction produces a slight rise in temperature of the liquid or an equivalent amount of heat is lost to the surroundings or there is any combination of the two. In any event, the liquid loses mechanical energy. Therefore it is customary to let

$$778(I_2 - I_1 + Q) = h_f, \tag{36}$$

where h_f is called the friction loss. Hence the energy equation for an incompressible fluid with negligible heat transfer is

$$\left(\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g}\right) - \left(\frac{p_2}{w} + z_2 + \frac{V_2^2}{2g}\right) = h_f - M. \tag{37}$$

This could also be derived from Eq. (34), for if v is constant, the integral of $v\,dp$ is $(p_1 - p_2)v = (p_1 - p_2)/w$. As has been observed before, this equation could be used to find M when all other quantities are known, but otherwise two sections would be chosen that did not include a pump or turbine between them. In this case M would not appear, and, unless a machine is involved, the symbol will not be included hereafter.

By the aid of Eq. (37), together with the equation of continuity, it is possible to compute the pressure variation along the length of a closed conduit, such as those shown in Figs. 54 and 55. The equation applies also to flow through a stream tube, which in the limit becomes a streamline. The pressure variation

determined by the energy equation applied to a streamline is shown in Fig. 56.

In practical cases the value of h_f may be relatively large, and again it may be very small. In the latter case it is often convenient to neglect it altogether, and therefore

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g} \tag{38}$$

or this may also be written as

$$\frac{p}{w} + z + \frac{V^2}{2g} = \text{constant.} \tag{39}$$

The equation in this form, which is for a *frictionless incompressible* fluid, was first given by Daniel Bernoulli in 1738 and is universally known as Bernoulli's theorem.[1]

[1] In classical hydrodynamics the energy equation is derived in a differential form by writing an equation for the forces on an elementary volume and using the principle that $F = ma = d(mV)/dt$. The forces considered are the pressure forces on the faces, the body force (gravity), and the forces due to viscosity. While it does not seem worth while to go through the entire derivation, the final results are of interest. Using the notation shown in Fig. 51, and writing the expression for the x component only, the acceleration is the left-hand side of the equation below, while the forces per unit mass are on the right-hand side. The x component of gravity is $-g\,\dfrac{dz}{dx}$; the term with p is the resultant of the pressure forces in the x direction; and the ν terms represent friction. The equation is then

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -g\frac{dz}{dx} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{3}\nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Similar expressions are obtainable for the y and z components. For one-dimensional motion, this becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{dz}{dx} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{4}{3}\nu\frac{\partial^2 u}{\partial x^2}.$$

This equation is entirely general, as it is for unsteady flow with friction. However, it is not possible to integrate it, and thus its usefulness is limited By omitting the viscosity term this can then be integrated along a streamline after multiplying by dx. Since $w = g\rho$, and replacing u by V, the result is

The equation of continuity and the energy equation in this final form show that for a stream of incompressible fluid, where the cross-sectional area is large and the streamlines widely spaced, the velocity is low and the pressure high; and where the cross-sectional area is small and the streamlines crowded together, the velocity is high and the pressure low. Hence a streamline diagram gives not only a picture of the velocity field but also one of the pressure field.

52. Significance of Head.—In Eq. (37) every term represents a linear dimension. Thus p/w is the height of a column of fluid of specific weight w which could produce the pressure p, z is a height by definition, while $V^2/2g$ is equivalent to the height that a body would fall in a vacuum under the action of gravity in order to acquire the velocity V. The elevation z is apparent in any specific case, but there may be no actual heights observable corresponding to either p/w or $V^2/2g$. While there may be such real physical heights in some cases, in others these dimensions are mere concepts. In the case of a compressible fluid where w is a variable, the consideration of p/w as a linear quantity is of doubtful value.

The term p/w is called *pressure head;* z is called *elevation,* or *potential, head;* and $V^2/2g$ is called *velocity head.* The sum of the three, $H = \frac{p}{w} + z + \frac{V^2}{2g}$, is called *total head.* It is obvious that h_f must also be a linear quantity, and it is called *friction head,* or *lost head.* Since it is a linear quantity; and since it is obviously some function of V, it may be represented as $h_f = kV^2/2g$, where k is an *abstract* number but not necessarily constant for different values of V.

For a frictionless incompressible fluid, $H_1 = H_2$; but for any real liquid,

$$H_1 = H_2 + h_f, \tag{40}$$

which is merely a brief way of writing Eq. (37) for the case where

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$$\frac{1}{g}\int_0^x \frac{\partial V}{\partial t}\,dx + \int_0^x \frac{1}{w}\,\frac{\partial p}{\partial x}\,dx + z + \frac{V^2}{2g} = \text{constant}.$$

This last equation is for a frictionless compressible fluid with unsteady flow. If w is constant for an incompressible fluid, and if $\partial V/\partial t = 0$ for steady flow, this becomes Bernoulli's equation (39). A valuable feature of this equation is that it shows the nature of the term to be included in the other equations in case of unsteady flow.

$M = 0$. By Eq. (37) it can be seen that, if a pump were used, H_2 would be greater than H_1 in general; but from Eq. (40) it is obvious that, unless there is some external energy input, friction causes a decrease in H in the direction of flow.

The equation derived in Art. 49 is for the flow of a weight of fluid $wA\,ds$ in time dt. Although the weight terms were factored out of the resulting equation, each term really represents energy for that weight. Thus head in any of its forms represents not only a *linear* dimension; it also represents *energy per unit weight;* and, if the time element is considered, it represents *power per unit rate of discharge.* That is, in specific units, head is not only in *feet;* it is also in *foot-pounds per pound* of fluid and in *foot-pounds per pound per second.*

With this latter interpretation of the meaning of the term *head* the expression for power is then

$$P = WH. \tag{41}$$

In the use of this equation H may be given any value of head for which the corresponding power is desired. For instance, if h_f is used in the equation, the result will be the value of the power consumed in friction. As an illustration, suppose that the surface of a lake is 500 ft. above the site for a powerhouse and that the lake is capable of furnishing a supply of 200 cu. ft. of water per sec. In this case $H = z = 500$ ft., and then $P = 62.4 \times 200 \times 500 = 6{,}240{,}000$ ft.-lb. per sec., or 11,340 hp., which is the power available. Again, suppose that a nozzle discharges 50 lb. of water per sec. in a jet with a velocity of 120 ft. per sec. In this case $H = V^2/2g = 224$ ft. Then

$$\frac{50 \times 224}{550} = 20.3 \text{ hp.}^1$$

[1] The expression "power equals force applied times velocity of the point of application of the force" cannot be used in either of the preceding cases, because it has no physical significance, since there is no force applied to anything, nor is there any point of application. In the case of a jet the force that it might exert would depend upon what happened when it struck an object, and the power produced would depend upon the velocity of the object but not that of the jet. But the available power of the jet is a definite quantity, no matter what it acts upon or whether it ever acts upon anything.

EXAMPLES

100. In Fig. 60 assume that a liquid flows at the rate of 6 cu. ft. per sec. from A to C, that friction loss is negligible from A to B, but that from B to C it is 0.1 $V_B^2/2g$. Find the pressures at A and C.

Ans. 74.6 ft., 67.25 ft.

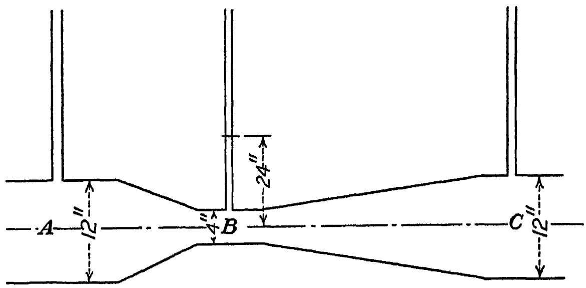


Fig. 60.

101. In Fig. 61 the pipe is of uniform diameter. The pressure at A is 20 lb. per sq. in., and at B is 30 lb. per sq. in. In which direction is the flow, and what is the loss in head if the liquid has a specific weight of (*a*) 30 lb. per cu. ft. and (*b*) 100 lb. per cu. ft.? *Ans.* 18 ft., 15.6 ft.

102. A pipe line conducts water from a reservoir to a powerhouse the elevation of which is 800 ft. lower than that of the surface of the lake. The water is delivered at a velocity of 220 ft. per sec. by a jet 10 in. in diameter. Find the power of the jet and the power lost in friction in the pipe line.

Ans. 10,250 hp., 650 hp.

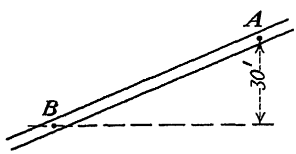


Fig. 61.

103. A pump lifts water at the rate of 5 cu. ft. per sec. to a height of 1,000 ft. It requires 850 hp. to run the pump of which 10 hp. is used in the bearings, the rest going into the water. Assuming no heat transfer, and the specific heat of water to be unity, what will be the rise in temperature of the water due to friction within pump and pipe line?

Ans. 0.618°F.

104. If in Fig. 52 the velocity of the undisturbed field is 20 ft. per sec., velocities on the surface on radii from the source at angles of 0, 30, 60, and 90 deg. are 0, 10.2, 18.6, and 23.7 ft. per sec., respectively. Assuming frictionless flow, compute pressures at those points relative to the pressure in the undisturbed field. *Ans.* 6.22, 4.60, 0.85, and −2.51 ft.

53. Pressure in Fluid Flow. *a. Pressure over Section.*—Strictly speaking, the energy equations that have just been derived apply to flow along a single streamline. The equations may, however, be used for streams of large cross-sectional area by taking average values, as has been explained in Arts. 41 and 48

in connection with the velocity. We shall now consider the question of the average pressure.

Referring to Fig. 62, consider a small vertical prism of a stream which is flowing in a horizontal direction. The vertical forces are the pressures on the top and the bottom of the prism and the force of gravity, which is the weight of the prism. All velocities and accelerations are horizontal and do not concern the vertical forces. Hence, writing an equilibrium equation, we have $p_1a + waL = p_2a$, where L is the height of the prism and a its cross-sectional area. From this we get $p_2 - p_1 = wL$, which is similar to Eq. (2). That is, in a plane normal to the velocity, the pressure varies according to the hydrostatic law. The average pressure is then the value of the pressure at the center of gravity of the area, as may be seen from Eq. (6). However, since $\frac{p}{w} + z$ is constant over the entire sectional area, it is immaterial what point is selected, provided the pressure and the elevation both apply to the same point.

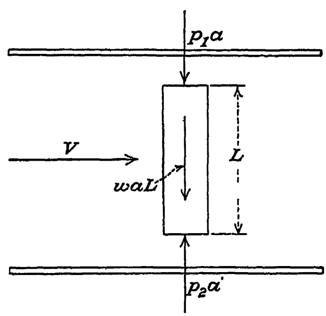


FIG. 62.

On a horizontal diameter through a pipe the pressure is then constant. Since, in many cases, the velocity is much higher in the center than near the pipe walls, it follows that H is also higher in the center. This is possible because the streamlines near the wall lose more energy in friction than those near the center of the stream. This consideration emphasizes the fact that the energy equation should be applied along the same streamline or the same stream but not between two streamlines any more than between two streams in different channels.

b. Negative Pressure.—In Art. 16 it has been stated that a true negative pressure for a liquid is impossible, since liquids can sustain only slight tensile stress. Excluding such phenomena as surface tension and capillarity, the least pressure that can be obtained with a body of liquid is its vapor pressure, which depends upon its temperature. Values of vapor pressure for water can be obtained from steam tables, since they are the saturation pressures corresponding to the temperatures. Values for other

vapors can be obtained from vapor tables so far as they exist. In hydraulics, engineers usually employ gage pressures, and the minimum value of the gage pressure that is possible is $p = p_v -$ barometer pressure, p_v indicating the vapor pressure, and all being expressed in the same units.

From Eq. (39) it appears that if z or $V^2/2g$ or both are increased sufficiently, the value of p/w may be reduced indefinitely. However, when the pressure has been reduced to the minimum value stated above, no further reduction can take place, as the liquid then boils or vaporizes, and we have the phenomenon that in flow is called *cavitation.* The physical conditions upon which the continuity and energy equations are based are no longer fulfilled, and hence the equations no longer apply. In fact, since water almost invariably contains dissolved air which will be liberated before the vapor pressure is reached, it is impossible to reduce the pressure to the vapor pressure without getting into difficulty.

c. Dynamic Pressure.—The center streamline in Fig. 52 shows that the velocity becomes zero at the *stagnation point* where the streamline contacts the body. If the pressure be taken as zero at some distance away, where the velocity is V, while the pressure at the stagnation point is indicated by p, then applying Eq. (35) to these two points,

$$0 + 0 + \frac{V^2}{2g} = \frac{p}{w} + 0 + 0,$$

from which

$$\frac{p}{w} = \frac{V^2}{2g}. \tag{42}$$

This is called the *dynamic pressure,* or the *velocity pressure.* It is the value of the pressure, above that of the so-called static pressure, that is obtained by dynamic means due to stopping a flow of velocity V.

54. Hydraulic Gradient.—If a piezometer tube be erected at B in Fig. 63, the liquid will rise in it to some height BB' equal to the pressure head existing at that point. If the lower end of the pipe were closed so that no flow could occur, the height of this column would evidently be BM. The drop from M to B' which is found when flow takes place is due to two factors, one of these

being that a portion of the pressure has been converted into the velocity head which the liquid has at B and the other that there has been a loss of head through friction between A and B.

If a series of piezometers were erected along the pipe line, the water would rise in them to various levels. The line drawn through the summits of such an imaginary series of liquid columns is called the *hydraulic grade line*, or the *hydraulic gradient*. It is seen that this line is an indication of the pressure variation along the pipe. Thus, at any point the vertical distance from the pipe line to the hydraulic gradient is the pressure at that point. Since at C this distance is zero, it follows that at C the pressure is atmospheric. And at D the line is below the pipe, indicating that at the point in question the pressure is below that of the atmosphere and is equal to $-DN$. The advantage of the construction of the

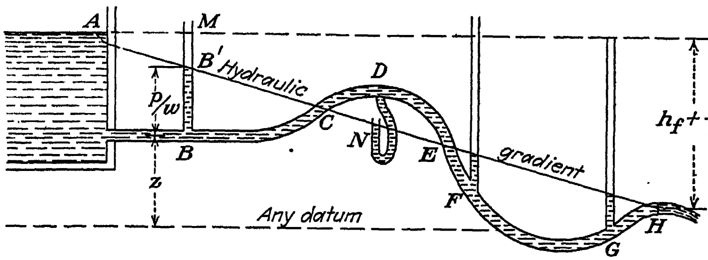


FIG. 63.

hydraulic gradient is that it gives a very clear picture of the pressure variation along a pipe line. Also, in practical applications the profile of a proposed pipe line should be drawn to scale. Then by computing a few points only, the hydraulic gradient can be drawn, and from it the pressures at all points can be readily measured.

The vertical distance from the hydraulic gradient to the level of the surface at A represents $h_f + V^2/2g$. Hence the position of the hydraulic gradient is independent of the position of the pipe line. Thus it is not always necessary to compute pressures at various points in order to plot the gradient. Instead, values of $h_f + V^2/2g$ may be laid off below the proper horizontal line, and this procedure is often more convenient. It is usually necessary to locate only a very few points; and often only two, the terminal points, are sufficient. For example, if Fig. 63 represents the profile of a pipe of uniform diameter drawn to scale, the

hydraulic gradient can readily be drawn as follows. At the intake to the pipe there will be a drop below the level of the surface which should be laid off equal to the sum of $V^2/2g$ plus the entrance loss.[1] At H the pressure is known to be zero gage pressure, and hence the gradient must pass through the end of the pipe. In the case shown, the hydraulic gradient is practically a straight line and hence may be drawn at once from these two points. The location of other points, as B', may be computed if desired. In the case of a long pipe line the velocity may be such that the drop in the gradient at the entrance is very small, and hence the error is very slight if the gradient is drawn as a straight line from the surface of the liquid above the intake to the lower end of the pipe.

The hydraulic gradient is not necessarily a straight line. For a pipe of uniform diameter it will be a straight line only if the pipe itself is straight. If the pipe is of uniform diameter, the drop in the hydraulic gradient along its length is then a measure of the loss of head, and this will be proportional to the horizontal distances in the figure only when the latter, in turn, are proportional to the actual lengths of pipe. But for ordinary amounts of curvature the hydraulic gradient will deviate but very little from a straight line. Of course, if there are losses of head aside from those due to ordinary pipe friction, there will be abrupt drops in it, and any variations in velocity head due to changes in diameter affect the hydraulic gradient.

It may be seen that, if the velocity head is constant, the drop in the hydraulic gradient between any two points is the measure of the loss of head between those two points. And the slope of the gradient is a measure of the rate of loss. Thus in Fig. 64 the rate of loss is much less in the larger pipe than in the smaller one. If the velocity changes, the hydraulic gradient might actually rise in the direction of flow, as may be seen in both Figs. 64 and 65. Additional illustrations of the hydraulic gradient for other cases are to be seen in Chaps. VIII and IX.

It is sometimes instructive to represent not only the variation of pressure head but also the variation of total head. If any arbitrary datum plane is assumed, the vertical distance from it to any point in the pipe represents the elevation head for that point. And the vertical distance from this point to the hydraulic gradient

[1] This and other details are explained more fully in Chap. VIII.

represents the pressure head. Hence, the vertical distance from the datum plane to the hydraulic grade line represents the sum of pressure head plus elevation head. If to this is added the velocity head, a curve is obtained, as shown in Fig. 65, the ordinates of which represent total head or energy. And, as in the case of the hydraulic gradient, the location of this total head curve is independent of the position of the pipe and may be

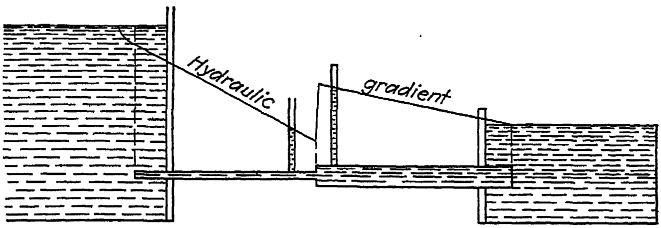


FIG. 64.

plotted by laying off values of loss of head below a horizontal line. The particular one shown, plotted from experiments made by the author, shows that the chief loss of head occurs in the diverging portion and just beyond the section of minimum diameter.

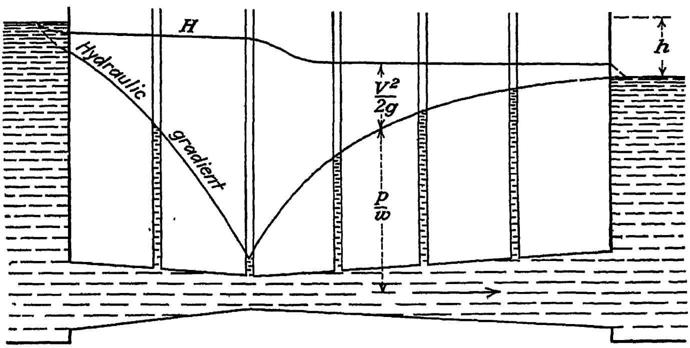


FIG. 65.

The total loss of head between the two tanks is h, and both entrance and discharge losses are here represented.

As shown in Fig. 65, for example, the hydraulic gradient or pressure gradient, as it might be called, may either fall or rise, but the curve for H, which might be called the *total energy gradient*, must fall continuously in the direction of flow, unless energy is received from some external source.

It may be observed that the hydraulic gradient in all cases represents what would be the free surface, if one could exist and maintain the same conditions of flow.

55. Method of Solution.—For the solution of problems of fluid flow we have two fundamental equations, the equation of continuity (18) or (19), and the energy equation in one of the forms from (29) to (39) inclusive. In most cases the following procedure may be employed:

1. Choose a datum plane through any convenient point.

2. Note at what sections the velocity is known or assumed. If at any point the cross section is great as compared with its value elsewhere, the velocity will be so small that the velocity head may be disregarded.

3. Note at what points the pressure is known or assumed. In a body of liquid at rest with a free surface the pressure is known at every point. The pressure in a jet is the same as that in the medium surrounding the jet.

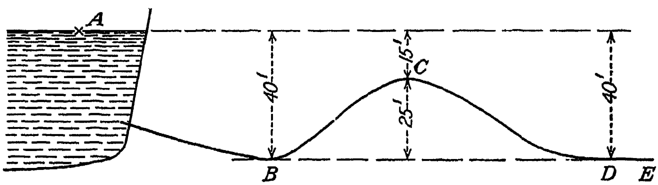


Fig. 66.

4. Note if there is any point where the three items of pressure, elevation, and velocity are known.

5. Note if there is any point where there is only one unknown quantity.

It is generally possible to write Eq. (37) between two points such that they fulfill conditions (4) and (5), respectively. Then the equation may be solved for the one unknown. If it is necessary to have two unknowns, then Eq. (37) must be solved simultaneously with Eq. (19). The procedure is best shown by applications such as the following:

In Fig. 66 water flows from reservoir A through pipe BCD, which is 6 in. in diameter. The diameter of the stream discharging freely into the air at E is 3 in. Assume that the loss of head in friction in any length of pipe may be represented as

$h_f = kV^2/2g$, where k depends upon the length of the pipe and other factors, which will be explained later. Suppose that the roughness of the pipe and the lengths between the various points are such that the values of k from the reservoir to B, from B to C, from C to D, and from D to E are 2, 4, 4, and 1, respectively, or the total loss in the entire pipe is $11V^2/2g$. Let it be required to find the pressure at C when flow takes place.

At C there are both an unknown pressure and an unknown velocity; hence Eq. (37) cannot be applied immediately, as one equation is capable of determining only one unknown. Let the procedure outlined then be followed. The location of a datum plane is immaterial in the solution of the problem, but it is usually convenient to take it through the lowest point in the figure and thus avoid negative values of z. Therefore let a datum plane through E be assumed. In the reservoir it is found that the velocity is negligible because of the large area as compared with the area of the pipe. At a point A on the surface of the water the pressure is found to be atmospheric, which is also the case with the stream at E. Thus whatever the pressure of the atmosphere may be, its effects can easily be shown to balance out, and therefore it is neglected altogether. Hence at A it is found that everything is known, while at E the velocity head is the only unknown.

It is immaterial whether the point A is on the surface or not. Since the fluid in the reservoir has a negligible velocity, there can be no friction losses in any case, and hence H has the same value at all points. A point at some distance y below the surface would have lost that much in elevation but would also have gained a pressure head of y. Therefore, in a reservoir,

$$\frac{p}{w} + z = \text{constant.}$$

That is, for any point in the reservoir,

$$H = \frac{p}{w} + z + \frac{V^2}{2g} = y + (40 - y) + 0 = 40.$$

It is also immaterial how far below the surface the intake for the pipe is located.

Applying Eq. (37) between points A and E, it is found that

$$H_A = 0 + 40 + 0,$$
$$H_E = 0 + 0 + \frac{V'^2}{2g},$$
$$h_f = 11\frac{V^2}{2g}.$$

Now, V' is the velocity of the jet at E while V is the velocity in the pipe, but one may be replaced in terms of the other by Eq. (19). (It is seldom necessary to compute areas for this. It is both easier and more accurate to use the ratios of the areas, which means the ratios of the diameters squared.) Now, $V' = AV/A'$, where A' is the area of the jet. But $A/A' = (6/3)^2 = 2^2 = 4$. Hence $V' = 4V$ and $V'^2 = 16V^2$. Replacing V' by V and substituting in Eq. (37),

$$40 - \frac{11V^2}{2g} = \frac{16V^2}{2g}.$$

Thus, $V^2/2g = 40/27 = 1.48$ ft. One of the unknowns at C has now been determined.

Next, Eq. (37) may be applied between C and either A or E, since the value of H at either of the latter points is known. The value of the effective head at C is $H = \frac{p}{w} + 25 + 1.48$, while $h_f = 6V^2/2g = 6 \times 1.48 = 8.88$ ft. Now, from Eq. (37)

$$40 - 8.88 = H_C = \frac{p}{w} + 26.48.$$

Hence,

$$\frac{p}{w} = 4.64 \text{ ft.}$$

If the rate of discharge is also desired, it can easily be found. Since $V^2/2g = 1.48$, $V = \sqrt{2g1.48} = 8.022\sqrt{1.48} = 9.75$ ft. per sec. Hence, $q = 0.196 \times 9.75 = 1.91$ cu. ft. per sec.

It will always be found more convenient to consider the velocity head $V^2/2g$ as a single quantity and to carry it through and find its value, rather than to substitute the numerical value of $2g$ earlier in the solution. If desired, the velocity may be found from $V^2/2g$ as shown. Often, however, as in finding the pressure at C, it is $V^2/2g$ and not the value of V that is required.

EXAMPLES

105. Compute the pressures at B and D in Fig. 66. *Ans.* 35.6, 23.7 ft.

106. Suppose that all other data for Fig. 66 remain unchanged except the diameter at C. What will this diameter be if there is a vacuum of 20 in. of mercury at C? *Ans.* 2.86 in.

107. Suppose that the diameter at C in Fig. 66 remains 6 in. and all other data are likewise unchanged except the elevation of C. How far above E can C be placed to produce a vacuum of 20 in. of mercury?

Ans. 52.24 ft.

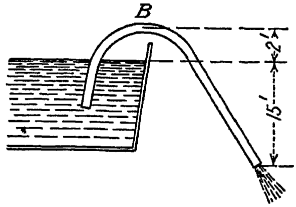


FIG. 67.

108. Assuming the flow to be frictionless in the siphon shown in Fig. 67, find the rate of discharge and the pressure head at B if the pipe is of a uniform diameter of 6 in. How would friction change these results?

Ans. 6.08 cu. ft. per sec., −17 ft.

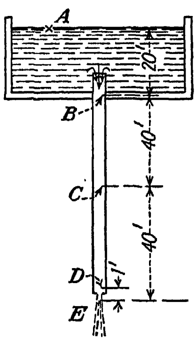


FIG. 68.

109. The diameter of the pipe in Fig. 68 is 4 in. and that of the stream discharging into the air at E is 3 in. Neglecting all losses of energy, what are the pressures at B, C, and D? (Velocity assumed negligible at A.)

Ans. −11.6, 28.4, 67.4 ft.

110. At B in Fig. 69 the diameter is 1 in., while the diameter of the jet in the air at C is 1.414 in. Neglecting friction losses, what are the values of the velocity and the pressure at B?

Ans. 45.4 ft. per sec., −24.0 ft.

111. What is the rate of discharge in the preceding problem, and what would it be if the tube and the jet from it were a uniform diameter of 1 in.?

Ans. 0.2475 cu. ft. per sec., 0.1238 cu. ft. per sec.

112. Suppose that the diameters at B and C are as specified in Prob. 110 but that the height of water in the tank is some unknown value h instead of 8 ft. Suppose also that the barometer pressure is 14.700 lb. per sq. in. and that the water in the tank is at a temperature of 100°F. At this temperature Table III gives $w = 62.00$, and steam tables give the vapor pressure as 0.946 lb. per sq. in. abs. What is the maximum value of h at which the tube will flow full? What would happen if h were increased above this value? *Ans.* 10.6 ft.

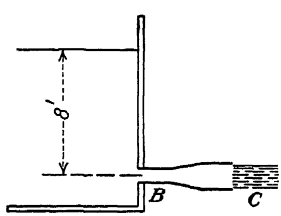


FIG. 69.

113. The pump P in Fig. 70 draws up liquid in the suction pipe with a velocity of 10 ft. per sec. Neglecting friction losses, find the maximum possible value of z for a barometer pressure of 14.700 lb. per sq. in. if the fluid is (a) water at 80°F. with a vapor pressure of 0.505 lb. per sq. in. and

specific weight of 62.22, (*b*) gasoline with a vapor pressure of 8.0 lb. per sq. in. and specific weight of 44. *Ans.* 31.34 ft., 20.34 ft.

114. Suppose that friction losses in the preceding problem were $2V^2/2g$; what would be the limiting values of z? *Ans.* 28.22 ft., 17.22 ft.

56. Flow in Curved Path.—The equation of energy applies to flow along a streamline. We shall now investigate conditions in a direction normal to the streamlines. In Fig. 71 is shown an elementary prism of width dr and sectional area da at a point where the radius of curvature of the path is r. The mass of this elementary volume is $\rho da\ dr$, and the normal component of its acceleration is V^2/r. Thus the centripetal force acting on it is $\rho\ da\ dr\ \frac{V^2}{r}$. Assume that as the radius varies from r to $r + dr$

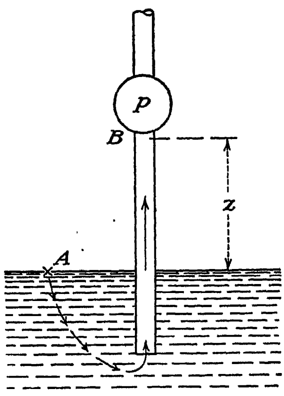


FIG. 70.

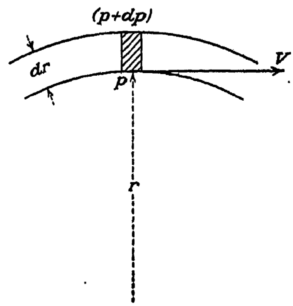


FIG. 71.

the pressure varies from p to $p + dp$. Then the resultant of the pressure forces in the direction of the center of curvature is $dp\ da$. Equating these two,

$$dp = \rho\frac{V^2}{r}\,dr. \tag{43}$$

This equation shows that when the flow is along a straight line, for which r is infinity, the value of dp is zero. That is, no difference in pressure can exist transverse to the flow in such a case. This has also been shown in Art. 53.

Since dp is positive if dr is positive, the equation also shows that the pressure increases as we proceed from the concave to the convex side of the stream, but the exact way in which it

increases depends upon the way in which V varies with the radius. In the next two articles will be presented two important practical cases in which V varies in quite different ways.

57. Forced Rotation.—A fluid may be made to rotate as a solid body without relative motion between particles, either by the rotation of the containing vessel or by stirring the contained fluid so as to make it rotate. Such a case is called a *forced vortex.* A common example is the rotation of liquid within the impeller of a centrifugal pump or the rotation of gas within the impeller of a blower or centrifugal compressor.

a. Cylindrical Vortex.—If the entire body of fluid rotates as a solid, $V = r\omega$, where ω is angular velocity. Inserting this value in Eq. (43), we have

$$dp = \rho\omega^2 r\,dr = \left(\frac{w}{g}\right)\omega^2 r\,dr.$$

Between any two radii r_1 and r_2 this integrates as

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{\omega^2}{2g}(r_2^2 - r_1^2). \tag{44}$$

If p_0 is the value of the pressure when $r_1 = 0$, this becomes

$$\frac{p}{w} = \frac{\omega^2}{2g}r^2 + \frac{p_0}{w}, \tag{45}$$

which is seen to be the equation of a parabola. In Fig. 72 it is seen that, if the fluid is a liquid, the pressure head p/w at any point is equal to z, the depth of the point below the free surface. Hence the preceding equations may also be written as

$$z_2 - z_1 = \frac{\omega^2}{2g}(r_2^2 - r_1^2) \tag{46}$$

and

$$z = \frac{\omega^2}{2g}r^2 + z_0, \tag{47}$$

where z_0 is the elevation when $r_1 = 0$. Equations (46) and (47) are the equations of the free surface, if one exists, or in any case are the equations for any surfaces of equal pressure, which are a series of paraboloids as shown by the dotted lines in Fig. 72 (a).

For the open vessel shown in Fig. 72 (*a*), the pressure head at any point is equal to its depth below the free surface. If the liquid is confined within a vessel, as shown in Fig. 72 (*b*), the pressure along any radius would vary in just the same way as if there were a free surface. Hence the two are equivalent. The case shown is purely ideal. Actually, owing to viscosity, the particles near the wall will lag, and thus the actual surface curve near the wall may fall a little below the paraboloid, as shown in Fig. 72 (*a*).

The axis of the open vessel must naturally be vertical. The axis of the closed vessel may be in any direction. However,

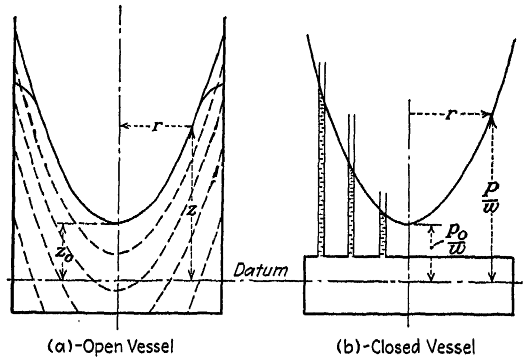


FIG. 72.—Forced vortex.

pressure varies with elevation as well as with the radius, and a more general equation, considering both of these factors at the same time, is

$$\frac{p_2}{w} - \frac{p_1}{w} + z_2 - z_1 = \frac{\omega^2}{2g}(r_2^2 - r_1^2). \tag{48}$$

Equation (44) is a special case when $z_1 = z_2$, and Eq. (46) is a special case when $p_1 = p_2$. If the axis of rotation of the closed vessel were horizontal, the paraboloid that represents the pressure would be somewhat distorted, as, at a given radius, the pressure at the top would be less than that at the bottom by the amount $z_2 - z_1$.

Inserting the value of p/w from Eq. (45), but letting $p_0/w = 0$, in the expression for total head, which is the constant in the Bernoulli equation (39), we have $H = \frac{p}{w} + \frac{V^2}{2g} = \frac{(r\omega)^2}{2g} + \frac{V^2}{2g}$.

Since p and V both increase or decrease together, which is just the opposite of the situation in linear flow, it is seen that H cannot be the same for different circular streamlines. In fact, since in this special case $V = r\omega$, it follows that $H = 2(r\omega)^2/2g$. That is, H increases as the square of r as we go from one circular streamline to the next.

b. Spiral Vortex.—So far the discussion has been confined to the rotation of all particles in concentric circles. Suppose that there is now superimposed a flow with a velocity having radial components, either outward or inward. If the height of the walls of the open vessel in Fig. 72 (*a*) were less than that of the liquid surface as shown, and if liquid were supplied to the center at the proper rate by some means, then it is obvious that liquid would flow outward. If, on the other hand, liquid flowed into the tank over the rim from some source at a higher elevation and were drawn out at the center, then the flow would be inward. The combination of this approximately radial flow with the circular flow will result in path lines that are some form of spirals.

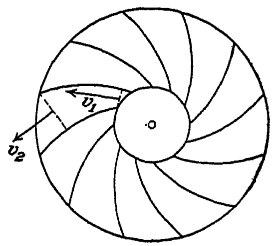


FIG. 73.

If the closed vessel in Fig. 72 (*b*) is arranged with suitable openings near the center and also around the periphery, and if it is provided with vanes, as shown in Fig. 73, it becomes either a centrifugal pump impeller or a turbine runner, as the case may be. These vanes constrain the flow of the liquid and determine both its relative magnitude and its direction. If the area of the passages normal to the direction of flow be a, then the equation of continuity fixes the relative velocities, since $q = a_1v_1 = a_2v_2 =$ constant. This relative flow is the flow as it would appear to an observer, preferably a camera, revolving with the vessel. The pressure difference due to this superimposed flow alone is found by the energy equation (38), neglecting friction losses, to be $\dfrac{p_2}{w} - \dfrac{p_1}{w} = \dfrac{v_1^2 - v_2^2}{2g}$.

Hence for the case of rotation with flow the total pressure difference between two points is found by adding together the pressure differences due to the two flows considered separately. That is,

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{\omega^2}{2g}(r_2^2 - r_1^2) + \frac{v_1^2 - v_2^2}{2g}. \tag{49}$$

Of course, friction losses will modify this result to some extent. It is seen that Eq. (44) is a special case of Eq. (49) when $v_1 = v_2$ either when both are finite or when $v_1 = v_2 = 0$.

For a forced vortex with spiral flow, energy is put into the fluid in the case of a pump and extracted from it in the case of a turbine. In the limiting case of zero flow, when all path lines become concentric circles, energy input from some external source is still necessary for any real fluid in order to maintain the rotation. Thus a forced vortex is characterized by a transfer of mechanical energy from an external source and a consequent variation of H as a function of the radius from the axis of rotation.

EXAMPLES

115. A closed vessel 12 in. in diameter completely filled with fluid is rotated at 1,700 r.p.m. What will be the pressure difference between the circumference and the center in feet of the fluid and in pounds per square inch if the fluid is (*a*) air with an average specific weight of 0.075 lb. per cu. ft., (*b*) water at 68°F., (*c*) mercury at 68°F. (see Table III, page 12)?

Ans. 123.2 ft., 0.0642 lb. per sq. in., 53.4 lb. per sq. in., 725 lb. per sq. in.

116. An open cylindrical vessel partially filled with water is 3 ft. in diameter. What speed of rotation about its axis would cause the water at the periphery to be theoretically 4 ft. higher than the water surface at the center? What would be the necessary speed if the fluid were mercury?

Ans. 102 r.p.m.

117. In Fig. 73 suppose that the vanes are all straight and radial, that $r_1 = 0.25$ ft., $r_2 = 0.75$ ft., and that the height perpendicular to the plane of the figure is $b = 0.2$ ft. = constant. (Then $a = 2\pi rb$.) If the speed is 1,200 r.p.m. and the flow is 7.54 cu. ft. per sec., find the difference in pressure between the outer and the inner circumference, neglecting friction losses. Does it make any difference whether the flow is outward or inward?

Ans. 130.8 ft.

58. Free Vortex.—A free vortex is one in which there is no expenditure of energy from an outside source and therefore, neglecting friction, H is constant throughout. Some examples are a whirlpool in a river, flow around a curve in a channel, the rotary flow that usually arises in a shallow vessel when liquid flows out through a hole in the bottom, and the flow in a centrifugal pump case just outside the impeller or in a spiral turbine case approaching the guide vanes.

a. Cylindrical Vortex.—In mechanics, force equals mass times acceleration, or force equals time rate of change of momentum,

from which the moment of the resultant force equals time rate of change of angular momentum. In the case of a free vortex, since no torque is applied to the fluid by an external agency, it follows that, neglecting friction, the angular momentum is everywhere the same.

The angular momentum of a particle of mass m rotating in a circular path of radius r with velocity V is mrV. Therefore, $rV = C =$ constant, where the value of C must be obtained by knowing the value of V at some definite value of r. Inserting $V = C/r$ in Eq. (43), we obtain

$$dp = \rho\frac{C^2}{r^2}\frac{dr}{r} = \frac{w}{g}\frac{C^2}{r^3}\,dr.$$

Between any two radii r_1 and r_2 this integrates as

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{C^2}{2g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = \frac{V_1^2}{2g}\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]. \qquad (50)$$

If there is a free surface, the pressure head p/w at any point is equal to the depth below the surface. Also, at any radius the pressure varies in a vertical direction according to the hydrostatic law. Hence, this equation is merely a special case where $z_1 = z_2$.

Since $H = \frac{p}{w} + z + \frac{V^2}{2g} =$ constant, it follows that at any radius r

$$\frac{p}{w} + z = H - \frac{V^2}{2g} = H - \frac{C^2}{2gr^2} = H - \frac{V_1^2}{2g}\left(\frac{r_1}{r}\right)^2. \qquad (51)$$

Assuming the axis to be vertical, the pressure along the radius can be found from this equation by taking z constant; and for any constant pressure p, values of z, determining a surface of equal pressure, can be found. If p is zero, the values of z determine the free surface, if one exists.

Equation (51) shows that H is the asymptote approached by $\frac{p}{w} + z$ as r approaches infinity and V approaches zero. On the other hand, as r approaches zero, V approaches infinity, and $\frac{p}{w} + z$ approaches minus infinity. Since this is physically impossible, the free vortex cannot extend to the axis of rotation. In reality, as high velocities are attained as the axis is

approached, the friction losses, which vary as the square of the velocity, become of increasing importance and are no longer negligible. Hence, the assumption that H is constant no longer holds even approximately. The *core* of the vortex tends to rotate as a solid body, thus forming a forced vortex, as shown in the central part of Fig. 74(b).

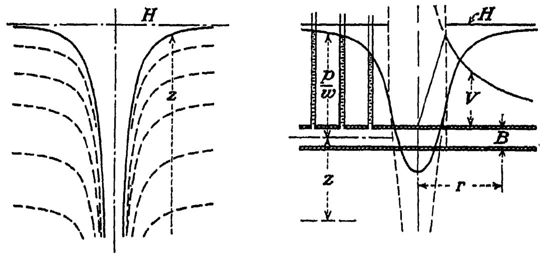


FIG. 74.—Free vortex.

The pressure and velocity distribution, as here described, apply to the case of flow in a curved channel such as in Fig. 75, where naturally the arc is only a portion of a circle. Pure rotation in complete concentric circles might be met with in the case sur-

FIG. 75.—Flow around bend.

rounding a centrifugal pump impeller when the pump is operating with the discharge valve closed. Figure 74(b) also pictures this, if the forced vortex in the center represents the action within the impeller while the free vortex surrounding it represents conditions in the case.

b. Spiral Vortex.—If a radial flow is superimposed upon the concentric flow previously described, the path lines will then be spirals. If the flow is out through a circular hole in the bottom of a shallow vessel, the surface of a liquid takes the form shown in Fig. 74 (*a*), with an air "core" sucked down the hole. If an outlet symmetrical with the axis is provided in the arrangement shown in Fig. 74 (*b*), we might have a flow either radially inward or radially outward. If the two plates shown are a constant distance B apart, the radial flow with a velocity V_r is then across a series of concentric cylindrical surfaces whose area is $2\pi rB$. Thus $q = 2\pi rBV_r =$ constant, from which it is seen that $rV_r =$ constant. Thus the radial velocity varies in the same way with r that the circumferential velocity did in the preceding discussion. Hence the pressure variation with the radial velocity is just the same as for pure rotation. Therefore the pressure gradient shown in Fig. 74 (*b*) applies exactly to the case of spiral flow, as well as to pure rotation.

In case B is not constant, then $V_r = \text{constant}/rB$, and the pressure variation due to the radial velocity will be somewhat different from that due to rotation alone. The true difference in pressure between any two points may then be determined by adding together the pressure differences caused by these two types of flow considered separately. This procedure is sometimes convenient when friction losses are to be considered and where the losses for the two types of flow are considered to vary in different ways. However, for the case where H is considered as constant for either type of flow, it is possible to combine V_r and V_u, the latter now being used to denote the circumferential velocity, to determine the true velocity V. The total pressure difference may then be determined directly by using this value of V.

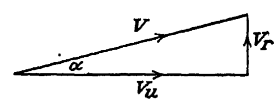


FIG. 76.

In Fig. 76 it is seen that, if both V_r and V_u vary inversely as the radius r, the angle α is constant. Hence for such a case the path line is the equiangular or logarithmic spiral. For the case where B varies, as it does in Fig. 74 (*a*), for instance, V_r will not vary in the same way as V_u, and hence the angle α is not a constant.

The plotting of such spiral path lines may be of practical value in those cases where it is desired to place some object in the

stream for structural reasons but where it is essential that the interference with the stream flow be a minimum. If the structure be shaped so as to conform to these path lines as nearly as possible, it will offer a minimum disturbance to the flow.

EXAMPLES

118. In Fig. 77 is shown a centrifugal pump with an impeller 12 in. in diameter surrounded by a casing which has a constant height B of 1.5 in. between a and b and then enlarges into the *volute* at c. Water leaves the impeller at a with a velocity of 60 ft. per sec. at an angle with the tangent of $\alpha = 15$ deg. (a) What will be the magnitude and direction of the velocity at b? (b) Neglecting friction, what will be the gain in pressure from a to b?

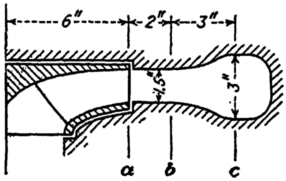


FIG. 77.

Ans. 45 ft. per sec., 15 deg., 24.5 ft.

119. For the case given in Prob. 118 find the magnitude and direction of the velocity at c. What is the gain in pressure between b and c? *Ans.* 31.9 ft. per sec., 7°37′, 15.65 ft.

120. In Fig. 78 is shown a spiral case for a large vertical-shaft turbine. In order to assist in supporting the weight of the generator on the floor above the turbine, columns are inserted in the casing in the form of vanes in a casting called, for no obviously good reason, a *speed ring*. These vanes should conform to the natural streamlines. (The guide vanes, which do direct the course of the water, are not shown but are just inside the inner circle of radius r_3. The runner, in turn, is in the very center, naturally.) In the figure shown, let $r_1 = 18$ ft., $r_2 = 8$ ft., $r_3 = 6$ ft., $A_1 = 200$ sq. ft., $B_2 = 3$ ft., $B_3 = 2.5$ ft., $\alpha_1 = 40$ deg. If the water enters the turbine case at (1) with a velocity of 8 ft. per sec., find the tangential and radial components of velocity at entrance to and exit from the speed ring. What should be the direction of the speed-ring vanes at entrance and at exit?

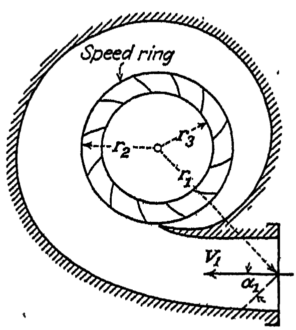


FIG. 78.

Ans. At (2) $V_{u_2} = 13.82$ ft. per sec., $V_{r_2} = 10.62$ ft. per sec., $\alpha_2 = 37.5°$.

121. In Fig. 79 is shown an open channel of rectangular cross section. In the straight portions of the channel the liquid is of a uniform depth y and, in the ideal case, has a uniform velocity of U. Then $q = byU =$ constant, and $H = y + \frac{U^2}{2g} =$ constant, if the datum plane be taken at the bottom. At a section in a bend in the channel with a radius of curvature to any path line of r, the liquid surface and the velocity will vary in the manner shown in Fig. 79 for the ideal case. Then $q = \int_{r_1}^{r_2} Vz\,dr$. Inserting in

this the expression for z from Eq. (51) and noting that $rV = r_1V_1 =$ constant, we obtain

$$q = byU = \left(y + \frac{U^2}{2g}\right)r_1V_1 \log_e \frac{r_2}{r_1} - V_1{}^3\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]\frac{r_1}{4g}.$$

From this the value of V_1 may really be determined by solution by trial. In reality, owing to friction effects, the velocity is often greater on the outside than it is on the inside of the curve, which is just the reverse of the ideal case illustrated in Fig. 79. However, the surface at the outside is higher than it is at the inner radius, just as is shown in Fig. 79, but the shape of the surface curve will be slightly different (see Fig. 75).

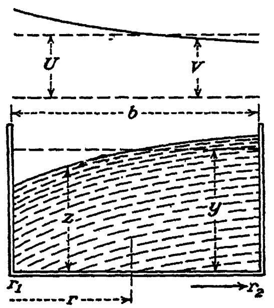


FIG. 79.

If in Fig. 79, $b = r_2 - r_1 = 10$ ft., $y = 5$ ft., $U = 10$ ft. per sec., $r_1 = 15$ ft., $r_2 = 25$ ft., find the depth of liquid in the channel at the inner and outer radii. *Ans.* 3.51 ft., 5.45 ft.

59. Dynamical Similarity.—In general, it is impossible to determine all the essential facts for a given fluid flow by pure theory, and hence much dependence must be placed upon experimental investigations. In both hydraulics and aeronautics valuable results can be obtained at a minimum cost by studies made on small-scale models; and even in the case of experiments made on full-size apparatus it is desirable to be able to apply the results to other cases. One of the first requirements in order that two cases may be compared is that there be geometrical similarity; that is, the model and its prototype must be of the same shape, though they may differ in size.

However, geometrical similarity is not enough. The two systems must also be dynamically similar, which requires that the paths of fluid particles are also geometrically similar and that the velocities and accelerations are such that corresponding forces are in the same ratio. The forces may be described as those due to inertia, friction, gravity, elasticity, and surface tension. The last named does not enter into the cases that will be discussed in this book, and elasticity enters only in the case of compressible fluids. Since even gases may be treated as incompressible where pressure differences are relatively small or where the velocities are not too high, and for air this is about 200 m.p.h., it is seen that the first three are the most important.

a. Reynolds Number.—Considering inertia and friction forces alone, a factor can be obtained which is called Reynolds number. In 1882 Osborne Reynolds published the results of some of his experimental work which involved this quantity, but it was Lord Rayleigh nearly ten years later who developed the theory of dynamic similarity and generalized the theorem.

Since from Newton's law, $F = d(mV)/dt$, the inertia forces are proportional to $\rho L^3V/T = \rho L^2(L/T)V = \rho L^2V^2$, where L is any linear dimension. In Art. 5 it was seen that the friction force equals $\mu AV/y$ and hence is proportional to μLV. Therefore the ratio of inertia forces to friction forces is proportional to

$$R = \frac{L^2V^2\rho}{LV\mu} = \frac{LV\rho}{\mu} = \frac{LV}{\nu}. \tag{52}$$

Since R is the ratio of two forces, it is an abstract number. This can be verified by substituting any consistent set of dimensions in Eq. (52) and finding that they all cancel.

Therefore, if two systems are dynamically equivalent, they must both have the same value of Reynolds number R. For the same fluid in both cases, a model with small linear dimensions must be used with correspondingly large velocities. It is also possible to compare two cases with fluids that differ widely from each other, provided only that L and V are properly chosen so as to give the same value of R.

Systems in which we are concerned only with inertia and friction forces are such as the flow of an incompressible fluid in a closed pipe; the flow of an incompressible fluid in an open channel, provided wave action at the surface plays no part; the movement of a submarine submerged far below the surface; or the movement of an airship.

b. Froude's Number.—Considering gravity and inertia forces alone, a ratio obtained by similar methods to the preceding is called Froude's number. Since the force of gravity is proportional to $wL^3 = g\rho L^3$,

$$\text{Froude's number} = \frac{\rho g L^3}{\rho L^2V^2} = \frac{gL}{V^2}. \tag{53}$$

However, some engineers use $V/\sqrt{gL}$ for the criterion.

Illustrations of systems involving gravity and inertia are the wave motion set up by a ship on the surface of the water or, to some extent, the flow of a jet of water from an orifice.

A comparison of Eqs. (52) and (53) shows that the two ratios cannot be satisfied at the same time, since one requires that the velocity vary inversely as L, while the other requires it to vary directly as $\sqrt{L}$. In a given case the one would be chosen that applied to the forces of predominating influence, and the results modified to allow for the effects of the other force. Thus, in the case of a ship on the surface, resistance to its motion due to the surface waves set up may be more important than the skin friction due to the viscosity of the water.

c. Acoustic Velocity Ratio.—Where compressibility enters, a third ratio is found to be V/C, where C is the velocity of a sound wave in the medium in question. In reality, a certain coefficient, for example, may be a function of LV/ν, gL/V^2, and V/C simultaneously. Thus its values would be represented by a family of curves and not by a single curve only.

In *a* and *b* the length L must be some linear dimension that is physically significant in the flow, such as the diameter of a circular pipe, the diameter of an orifice, the length of a ship.

For circular pipes the dimension L is sometimes taken as the radius, but the diameter is more commonly employed. One is just as logical as the other, but the resulting value of R will be twice as much for the diameter as for the radius. In reading the literature where R is involved, it is necessary to observe which convention has been followed.

For a circular cross section it is often convenient to express Eq. (52) in certain other forms. Thus if d is the diameter in feet and D the diameter in inches, and, since $q = V\pi D^2/576$ and $w = g\rho = 32.17\rho$, we obtain

$$R = 0.4749\frac{wq}{D\mu} = 15.276\frac{q}{D\nu}.$$

In Eq. (52) any consistent system of units may be used, but in this last form μ must be in pounds second per square foot and ν must be in square feet per second.

The preceding expression is satisfactory for liquids, but for gases it is more convenient to employ weight rather than volume so as to be independent of the pressure. Also, absolute viscosity is usually given in poises or centipoises. Since $W = wq$, and making the conversion of viscosity units,

$$R = 22{,}740\,\frac{W}{D \times \text{centipoises}}.$$

60. PROBLEMS

122. In Fig. 70 let $z = 15$ ft. and at B let $V = 10$ ft. per sec. Also, assume that friction losses between A and B have the value $1.5V^2/2g$ and that the value is the same for either direction of flow. Find the pressure head at B (*a*) if the pipe is the suction pipe for a pump, (*b*) if the pipe is the draft tube for a turbine. *Ans.* -18.89 ft., -14.23 ft.

123. In Fig. 80 let $h = 100$ ft. Assume that the pipe is of uniform diameter and discharges freely into the air at (2) a stream which is the same diameter as the pipe. Assume also that the loss of head between (1) and (2) is given by $h_f = kV^2/2g$ and that in this case k is taken arbitrarily as 24. (*a*) Find velocity of flow and draw hydraulic grade line. (*b*) If there were no friction losses, what would be the velocity, and what would be the hydraulic gradient? *Ans.* (*a*) 16.05 ft. per sec.

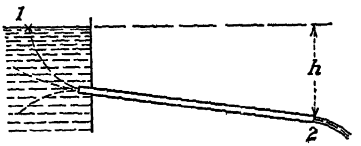


FIG. 80.

124. If in Fig. 52 the velocity of the undisturbed field is 20 ft. per sec., the velocities at the surface on radii from the source at angles of 0, 60, 120, 150 deg. are 0, 18.6, 25.2, 23.3 ft. per sec., respectively, for the ideal case of frictionless flow. For the ideal case, what will be the elevation of the surface of a liquid adjacent to these points on the body relative to the undisturbed surface? (This problem illustrates the way in which the water surface drops alongside a bridge pier or past the side of a moving ship.)
Ans. 6.22, 0.85, -3.65, -2.22 ft.

125. If in Fig. 56 the velocity of the undisturbed field is 20 ft. per sec., the velocities at the surface on radii from the source of angles of 0, 60, 120, 150 deg. are 0. 15, 22.8, 21.2 ft. per sec., respectively, for a frictionless fluid. For the ideal case what will be the pressures at these points in a horizontal plane containing the axis and relative to the pressure in the undisturbed field? If the body is an airship, what will be the pressures in a vertical plane? For air, what additional information is necessary in order that pressures may be expressed in pounds per square inch?
Ans. 6.22, 1.07, -1.88, -0.71 ft. of fluid.

126. In Prob. 125 assume that the body is a submarine with diameters at the four points mentioned of 0, 8, 13.86, 15.44 ft., respectively. If the submarine is submerged in the ocean with its axis 40 ft. below the surface, find the actual pressures at these four points along the top and along the bottom. Express answers in pounds per square inch. *Ans.* At 0 deg., 20.5 lb. per sq. in.; at 150 deg., 14 lb. per sq. in.; 20.9 lb. per sq. in.

127. The diameters of the suction and discharge pipes of a pump are 6 and 4 in., respectively. The discharge pressure gage at a point 5 ft. above the center line of the pump reads 20 lb. per sq. in., while the suction pressure gage records a vacuum of 10 in. of mercury at a point 2 ft. below the center line. If the pump delivers water at the rate of 0.9 cu. ft. per sec., find the power delivered to the water. *Ans.* 6.72 hp.

128. If all the data given above applied to a pump handling gasoline with a specific gravity of 0.75, what would be the power? *Ans.* 6.52 hp.

129. A pump circulates water at the rate of 2,000 g.p.m. in a closed circuit holding 10,000 gal. The net head developed by the pump is 115 ft., and the efficiency of the pump is 80 per cent. Assuming the friction in the bearings to be negligible and that there is no loss of heat from the system, find the temperature rise in the water in 1 hr. *Ans.* 2.2°F.

130. For the three-dimensional flow against a flat plate in Prob. 86, the velocity was found to be given by the equation $V^2 = a^2(4x^2 + y^2)$. Prove that the pressure on the plate at any radius y is given by the equation

$$\frac{p}{w} = \frac{U^2}{2g} - \frac{a^2y^2}{2g}.$$

131. Two horizontal circular plates are parallel to each other and a fixed distance apart. Between them flows a frictionless incompressible fluid with a velocity that is either radial or spiral. Prove that the total pressure on an annular ring is given by $\pi w[H(r_2^2 - r_1^2) - (2r_1^2V_1^2/2g) \log_e (r_2/r_1)]$.

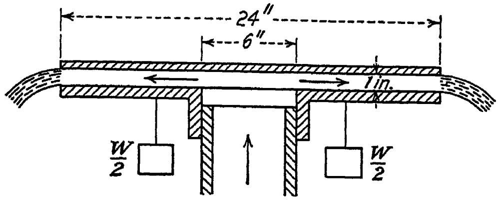


FIG. 81.

132. In Fig. 81 the upper circular plate is fixed while the lower annular plate is free to move vertically and is not supported by the pipe in the center. Water is admitted at the center at the rate of 2 cu. ft. per sec. and discharges into the air around the periphery; therefore the pressure at the 12-in. radius is atmospheric. What is the total weight W that will be supported by the annular plate, whose weight is 5 lb., if the distance between the two is maintained at 1 in.? *Ans.* 65.2 lb.

133. In Prob. 132 what is the pressure where the radius is 3 in.? *Ans.* −3.40 ft.

134. In Fig. 73 let the two radii be 6 and 18 in., respectively, and assume that the height B perpendicular to the plane of the paper varies in such a way that the cross-sectional area of the water passages is constant. Assume that the value of v is 35 ft. per sec. and that the centrifugal pump impeller runs at 600 r.p.m. Using the same method as in Fig. 53, plot the absolute path of the water for one or more assumed vane curves.

135. (*a*) What is the value of Reynolds number for the case of water at 60°F. flowing in a pipe 6 in. in diameter with a velocity of 4 ft. per sec. (see Table III)? (*b*) What would be the value of R if the fluid were air at 100 lb. per sq. in. abs. and 150°F. (see Fig. 2 and note that for air,

$$w = \frac{p}{53.34T} \text{ lb. per cu. ft.}\Big)?$$

Ans. 164,000; 65,000.

136. A ship 600 ft. long is to operate at a speed of 30 ft. per sec. What should be the corresponding speed for a model 10 ft. long?

Ans. 3.88 ft. per sec.

137. Suppose that the ship in Prob. 136 were to operate in ocean water whose viscosity was 1.2 centipoises and specific weight 64 lb. per cu. ft.; what would have to be the kinematic viscosity of the fluid used for the model if its Reynolds number as well as its Froude's number were the same as that of the prototype? *Ans.* 2.71×10^{-6} sq. ft. per sec.

138. An airplane model has linear dimensions that are one-fortieth those of the full-sized plane. It is desirable to test it in a wind tunnel with an air speed equal to the speed of the actual plane. If the Reynolds number is the same, what must be the pressure of the air in an enclosed tunnel?

Ans. 40 atmospheres.

139. In Fig. 80, assume that the pipe is initially full of water but that a valve at the lower end is closed. Assume that the friction loss may be expressed as $kV_2^2/2g$ and that k is constant for all values of V_2. If the valve were instantaneously opened, how long would it take for the velocity to attain its final equilibrium value? This is a problem in unsteady flow and requires the application of the equation obtained in the footnote of Art. 51. If V is the velocity at any point, while V_2 is the velocity at the end, $AV = A_2V_2$. The area is some function of the distance x along the pipe, or $A = f'(x)$; and therefore $V = V_2 f(x)$. Hence, $\frac{\partial V}{\partial t} = \frac{dV_2}{dt} f(x)$. Thus $\int_0^L \left(\frac{\partial V}{\partial t}\right) dx = \frac{dV_2}{dt}\int_0^L f(x)\, dx$. If the pipe is of a uniform diameter, as is the case assumed in this problem, $\int_0^L (\partial V/\partial t)\, dx = (dV_2/dt)L$. In reality, L would be slightly longer than the pipe in order to allow for some flow in the reservoir.

The pressure at the end of the pipe is atmospheric, and, if the datum plane be taken at (2), the differential equation reduces for this case to

$$\frac{L}{g}\frac{dV_2}{dt} + \frac{V_2^2}{2g} = h - k\frac{V_2^2}{2g}.$$

When equilibrium is attained and the velocity becomes constant at V_0, then $h = (1 + k)V_0^2/2g$. The equation may then be arranged as

$$\frac{dV_2}{V_0^2 - V_2^2} = \frac{(1 + k)\, dt}{2L}.$$

Integrating this equation, we obtain

$$t = \frac{L}{(1 + k)V_0} \log_e \frac{V_0 + V_2}{V_0 - V_2}.$$

If the length of the pipe is 1,000 ft., the equilibrium velocity 4 ft. per sec., and the value of k 24, how long would it take for the velocity to reach values of 0.95, 0.99, and 0.9999 times the equilibrium velocity? How long would it take to reach equilibrium? *Ans.* 36.6, 52.8, 99.0 sec.; infinite time.

CHAPTER VII

APPLICATIONS OF HYDROKINETICS

61. Measurement of Flow.—For the most part, the present chapter will deal with the applications of the principles of the dynamics of fluid flow to those devices whose primary use is that of the measurement of either velocity or rate of discharge. The ultimate standard in the determination of the rate of discharge is the measurement of either the volume or the weight of fluid in a known length of time; but it is obvious that such measurements are often impossible, as in the flow of a large river, for example, and that in many cases they are impractical. Therefore, it is customary to employ instruments or devices which are more convenient but which must either be calibrated or else a coefficient assumed for them on the basis of experimental work which has been reported on similar instruments. In order that this may be done intelligently, it is necessary to know the theory underlying each instrument. The more precisely the theory is known for a given device the more reliable becomes our application of the results of experimental work.

62. Orifices, Tubes, and Nozzles.—An *orifice* is an opening in the wall of a containing vessel. The principal restriction is that the thickness of the wall shall be only a small fraction of the diameter, or similar dimension, of the opening. A *standard* orifice is one in which the edge is a sharp *knife-edge,* as in Fig. 82 (*a*), or is a perfectly square corner in a thin plate, as in Fig. 82 (*b*). In both of these the fluid is in contact with a line only. They are called *standard orifices* because it is apparently possible for them to be duplicated so that one of them will give the same results as another of the same size. Other forms such as (*c*) and (*d*) would give different results, depending upon the thickness of the plate, the roughness of the surface of the opening, the radius of curvature of the upstream edge, etc.; hence each would require individual calibration if a high degree of accuracy were required.

A *tube* is a short pipe whose length is not more than two or three diameters. It differs from the standard orifice in that its length,

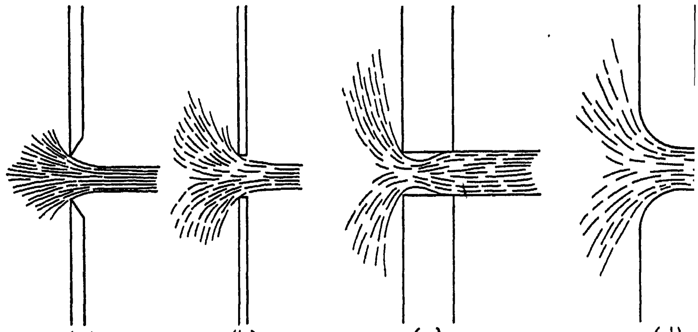


FIG. 82.—Types of orifices.

FIG. 83.—Jet from hydraulic giant washing out material for earth

parallel to the direction of flow, becomes a factor. There is no sharp distinction between the tube and the thick-walled orifices, such as those in (*c*) and (*d*) of Fig. 82. Tubes may be straight, converging, or diverging.

A *nozzle* may be defined as a converging tube. For a gas or a vapor, the nozzle may converge at first and then diverge again after attaining a section of minimum area because, as lower pressures are reached, the specific volume may increase faster than does the velocity [see Eq. (18)].

These three devices may be used to measure the rate of discharge, and again they may serve other purposes. Thus the

FIG. 84.—Jet from 7½-inch nozzle. (Head = 822 ft., velocity = 227.4 ft. per sec.) (*From a photograph by W. R. Eckart, Jr.*)

nozzle is often used to provide a .ty stream for fire fighting, for a service such as is sho 3, or for power in a steam turbine or Pelton water wheel, as in Fig. 84.

63. Definition of a Jet.—A jet is a stream issuing from an orifice, tube, or nozzle. It is not enclosed by solid boundary walls but is surrounded by a fluid whose velocity is less than its own. The two fluids may be different, or they may be of the same kind. A *free* jet is a stream of liquid surrounded by a gas and is therefore directly under the influence of gravity. A *submerged* jet is one of any fluid surrounded by a fluid of the same kind. Since it is buoyed up by this surrounding fluid, it is not directly under the action of gravity.

64. Jet Contraction.—Where the streamlines converge in approaching an orifice, as shown in Fig. 85, they continue to converge beyond the orifice until they reach the section x-y, where they become parallel. For the ideal case, this location is an infinite distance away; but in reality, owing to friction between the jet and the surrounding fluid, this section is usually found very close to the orifice. For a free jet with a high velocity or a submerged jet with any velocity, the friction with the surroundings is such that the jet slows down and diverges beyond x-y, as is seen in Fig. 84. The section x-y is then a section of minimum area and is called the *vena contracta.*[1]

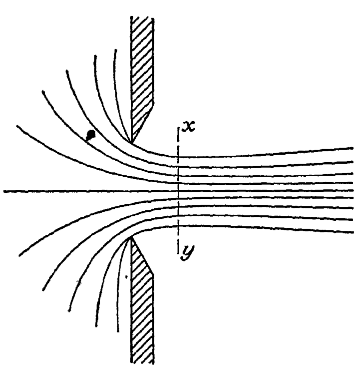


FIG. 85.—Jet contraction.

By jet velocity is meant the value of the average velocity at the vena contracta. The velocity profile for a jet is shown in Fig. 86, where it is seen to be very nearly constant except for a small annular ring around the outside. The average velocity V is thus only slightly less than the center or maximum velocity V_c. Since the streamlines are straight at the section x-y, it follows that at the vena contracta the pressure is constant across the diameter and, except for a minute increase due to surface tension, that this pressure must be equal to that in the medium surrounding the jet.[2]

In the plane of the orifice the cross-sectional area is greater than at x-y, and hence the average velocity must be less. Since

[1] Of course, if a free jet is discharged vertically downward, the acceleration due to gravity will cause its velocity to increase and the area to decrease continuously. Also, a free jet, which is initially horizontal, will describe a parabolic path under the action of gravity and with a vertical velocity component which is continually increasing. If the initial jet velocity is quite low, the friction between the liquid and the surrounding air or other gas may be insufficient to offset this acceleration due to gravity so that there may be no apparent section of minimum area. However, in such special cases, the vena contracta should be taken as the place where marked contraction ceases and before the place where gravity has increased the velocity to any appreciable extent above the true jet velocity.

[2] The *pressure* within a jet is totally different from the *force* that the jet is capable of exerting upon some object that it may *strike.*

the streamlines at this point are curved, centrifugal action will cause the pressure to increase from the edge of the orifice to the center, in accordance with Art. 56. Thus the pressure across the diameter will vary somewhat as shown in Fig. 86, and consequently, from the equation of energy, the velocity V' must also vary in the manner shown.

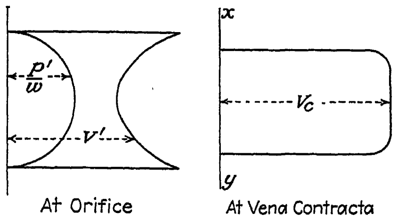


FIG. 86.—Pressure and velocity variation.

Since the contraction of the jet is due to the converging of the approaching streamlines, it may be prevented altogether by causing the particles of fluid to approach in an axial direction solely, as in the pipe in Fig. 87, which is an extreme case. It may also be prevented by a construction such as in (*c*) and (*d*) in Fig. 82.

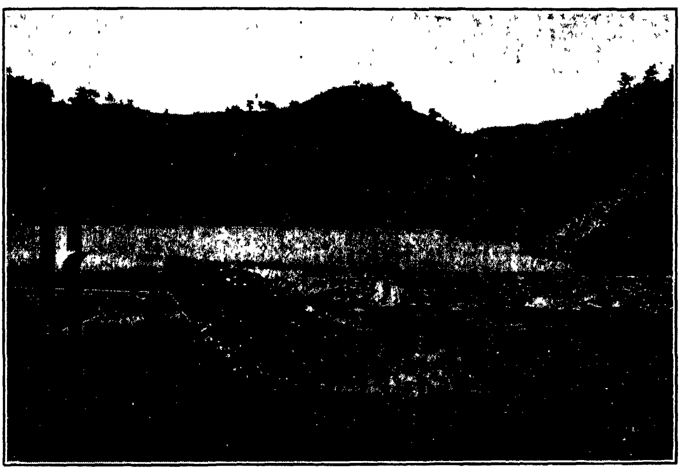


FIG. 87.—Discharge from end of a straight pipe. (Mixture of water, mud, and rocks building up the Calaveras earth dam.)

Also, if an orifice is placed too near the bottom or the side of a tank, the streamlines are prevented from converging to the full

extent, and thus the contraction is diminished. Roughness of the surface around the orifice reduces the velocity with which the fluid flows in from the side and also has an effect in reducing the amount of contraction.

65. Jet Coefficients. *a. Coefficient of Velocity.*—The velocity that would be attained if friction did not exist may be termed the *ideal velocity.*[1] It is practically the value of V_c in Fig. 86. Owing to friction the actual average velocity is less than this ideal value, and the ratio of the actual average velocity to the ideal velocity is called the coefficient of velocity.

b. Coefficient of Contraction.—The ratio of the area of a jet at the section of minimum area to the area of the orifice or other opening is called the coefficient of contraction.

c. Coefficient of Discharge.—The ratio of the actual rate of discharge to the ideal rate of discharge, if there were no friction and no contraction, may be defined as the coefficient of discharge.

In certain cases the discharge coefficient is the product of the other two. That is, $c_d = c_c \times c_v$; but this is not always true, as will be shown in Art. 84.

The coefficient of discharge may be obtained directly by measuring the flow of a jet by some standard method such as weighing the flow for a known time interval on scales or else observing the change in volume in a known time in a calibrated tank, if the fluid is a liquid. For a gas other means must be adopted, such as noting the change in pressure and temperature in a container of known volume from which the gas may flow.

The coefficient of velocity may be obtained by determining the velocity profile across a section by the aid of some instrument which measures velocity at a point. In the case of a liquid in a free jet, if the coordinates of the trajectory are measured, it is possible to compute the initial velocity by the principles of mechanics. In a free jet each particle may be considered as moving independent of any other, and thus each particle will describe the path followed by a body moving under the action of gravity and air resistance.

[1] This is frequently called "theoretical velocity" by others, but the author feels that this is a misuse of the word "theoretical." Any correct and sensible theory should allow for the fact that friction exists and affects the result. Otherwise, it is not theory but merely an incorrect hypothesis.

The coefficient of contraction may be obtained by measuring the diameter of the jet itself. If any two of the coefficients can be found, the third one is then determined.

66. Flow through Orifices, Tubes, and Nozzles.—In Fig. 88 let the energy equation (37) be written between points (1) and (2), assuming that the pressure is atmospheric, or the same at both points, and that the area of the vessel is so large that the velocity at (1) is negligible. Also, assume that the friction between the two points may be represented as $h_f = kV^2/2g$. Since

$$H_1 = 0 + h + 0 \text{ and } H_2 = 0 + 0 + \frac{V^2}{2g},$$

we have

$$h - \frac{kV^2}{2g} = \frac{V^2}{2g},$$

from which

$$V = \frac{1}{\sqrt{1+k}}\sqrt{2gh}. \quad (54)$$

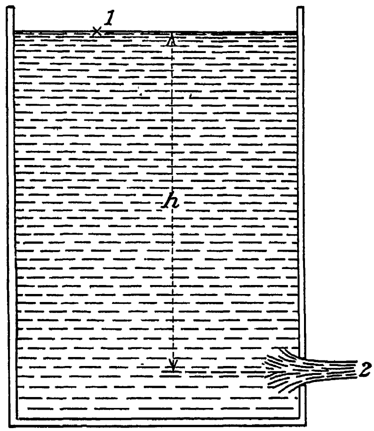


FIG. 88.

If there were no frictional resistance to flow, the value of k would be zero. Thus the *ideal* velocity is $V_i = \sqrt{2gh}$. Since the true velocity is obtained by multiplying the ideal velocity by the velocity coefficient,

$$V = c_v\sqrt{2gh}. \quad (55)$$

From this it follows that

$$c_v = \frac{1}{\sqrt{1+k}} \quad (56)$$

or

$$k = \frac{1}{c_v^2} - 1. \quad (57)$$

These two equations express the relation between the coefficient of velocity and the coefficient of loss.

If V and A denote the velocity and the area of the jet, respectively, while A_0 denotes the area of the orifice, it may be seen that

$$q = AV = (c_c A_0)(c_v\sqrt{2gh}) = c_d A_0 \sqrt{2gh}. \qquad (58)$$

In case the pressures at (1) and (2) are not the same, as might be the case, then in place of h in the preceding equations we should have $h + \frac{p_1}{w} - \frac{p_2}{w}$.

67. Submerged Orifice.—For a submerged orifice, as shown in Fig. 89, the following could be written for points (1) and (2): $H_1 = 0 + (h + y) + 0$ and $H_2 = y + 0 + V^2/2g$. This is based on the assumption that the pressure at (2) is equal to y. Since this term cancels out, it is seen that

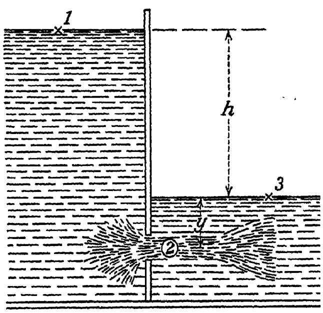


Fig. 89.

$$V = c_v\sqrt{2gh},$$

just as in the preceding case.

For a submerged orifice the streamlines and the coefficients of velocity and contraction are practically the same as in the case where there is a free jet. For heads under 10 ft. and for small orifices, the discharge may be slightly reduced, but for most cases the difference is negligible.

It is of interest to observe that, if the energy equation be written between (1) and (3), the result is

$$h_f = H_1 - H_3 = h.$$

Therefore, this is one case where it is impossible to assume an ideal condition involving zero friction loss, such as is possible between (1) and (2), for example. Likewise, between (2) and (3),

$$h_f = \left(y + \frac{V^2}{2g}\right) - y = \frac{V^2}{2g},$$

which expresses the fact that the velocity head is all dissipated between those two points.

68. Velocity of Approach.—In the preceding discussion it was assumed that the cross section of the vessel, from which the liquid issued, was so large that the velocity at (1) was negligible. In case this is not so, it is necessary to consider the velocity head

at (1). The velocity at this point is called *velocity of approach.*

The velocity of approach is usually negligible in all cases where the area at (1) is more than sixteen times that of the jet. Hence, it is generally necessary to consider it only in some cases of an orifice or a nozzle on the end of a pipe, such as is shown in Fig. 90. For sections (1) and (2), $H_1 = (p_1/w) + 0 + (V_1^2/2g)$ and $H_2 = 0 + 0 + (V_2^2/2g)$. If $h_f = kV_2^2/2g$,

$$\frac{p_1}{w} + \frac{V_1^2}{2g} - \frac{kV_2^2}{2g} = \frac{V_2^2}{2g}.$$

Therefore,

$$V_2 = \frac{1}{\sqrt{1+k}}\sqrt{2gH_1} = c_v\sqrt{2g\left(\frac{p_1}{w} + \frac{V_1^2}{2g}\right)}. \tag{59}$$

If the velocity at (1) were known in some way, this equation could

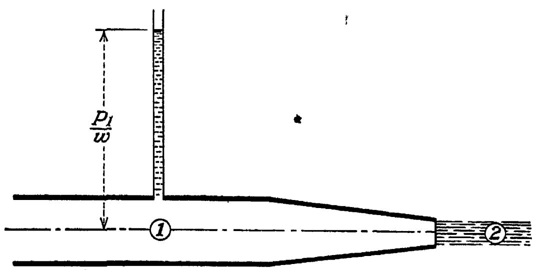


FIG. 90.

be used directly to find the velocity of the jet. Even if V_1 is known only approximately, the error in using such a value may not be great, since $V_1^2/2g$ is usually small compared with p_1/w. However, V_1 can be eliminated by the continuity equation $A_1V_1 = A_2V_2 = c_cA_0V_2$. Using this relation in Eq. (59), there is obtained, if $c_c \times c_v = c$,

$$V_2 = \frac{c_v}{\sqrt{1 - (c_vA_2/A_1)^2}}\sqrt{2g\frac{p_1}{w}} = \frac{c_v}{\sqrt{1 - (cA_0/A_1)^2}}\sqrt{2g\frac{p_1}{w}}. \tag{60}$$

Multiplying V_2 by $A_2 = c_cA_0$, the result is

$$q = \frac{cA_0}{\sqrt{1 - (cA_0/A_1)^2}}\sqrt{2g\frac{p_1}{w}} = \frac{cA_0}{\sqrt{1 - c^2(D_0/D_1)^4}}\sqrt{2g\frac{p_1}{w}}, \tag{61}$$

where D_1 is the diameter of the pipe and D_0 the diameter of the

orifice. Comparing this with Eq. (58), it is seen that

$$\sqrt{1 - c^2(D_0/D_1)^4}$$

is the factor that corrects for velocity of approach.[1]

EXAMPLES

140. Water issues from a circular orifice under a head of 25 ft. The diameter of the orifice is 3 in. If the discharge is found to be 355 cu. ft. in 5 min., what is the coefficient of discharge? If the diameter at the vena contracta is 2.36 in., what is the coefficient of contraction, and what is the coefficient of velocity? *Ans.* 0.602, 0.620, 0.97.

141. A jet discharges from an orifice in a vertical plane under a head of 9 ft. The diameter of the orifice is 2 in., and the measured discharge is 0.315 cu. ft. per sec. The coordinates of the center line of the jet are 8.14 ft. horizontally from the orifice and 2.0 ft. below the center of the orifice. Find the coefficients of discharge, velocity, and contraction.

Ans. 0.600, 0.96, 0.625.

142. The velocity of water in a pipe 6 in. in diameter is 12 ft. per sec. At the end of the pipe is a nozzle whose velocity coefficient is 0.98. If the pressure in the pipe is 10 lb. per sq. in., what is the velocity of the jet? What is the diameter of the jet? What is the rate of discharge?

Ans. 39.6 ft. per sec., 3.3 in., 2.35 cu. ft. per sec.

143. A jet of water 2 in. in diameter is discharged through a nozzle whose velocity coefficient is 0.98. If the pressure in the pipe is 10 lb. per sq. in. and the pipe diameter is 6 in., what is the velocity of the jet? What is the rate of discharge? *Ans.* 38.1 ft. per sec., 0.83 cu. ft. per sec.

144. At the end of a pipe which is 4 in. in diameter is an orifice 3 in. in diameter. The pressure in the pipe is 8 lb. per sq. in., and the fluid has a specific weight of 60 lb. per cu. ft. If the velocity coefficient is 0.97 and the discharge coefficient 0.90, find the jet velocity and the rate of discharge.

Ans. 39.6 ft. per sec., 1.745 cu. ft. per sec.

69. Values of Coefficients. *a. Velocity Coefficient.*—For sharp-edged circular orifices and for properly shaped nozzles with smooth walls the coefficient of velocity ranges from about 0.95 to about 0.994. The smaller values are found with small orifices and low heads, and the larger values with larger sizes and higher heads. The coefficient increases more with increasing size than it does with increasing head.

For certain tubes and other devices the velocity coefficient may be much smaller than the values for standard orifices and nozzles. Thus the orifice shown in Fig. 82 (*c*) will have a relatively low velocity coefficient due to the disturbed flow resulting

[1] Discussion of this case will be continued in Art. 84.

from the contraction of the stream and its subsequent reexpansion. Rounding the entrance edge so as to avoid this, as in Fig. 82 (*d*), improves the conditions very greatly. Typical values for a few cases are shown in Fig. 91.

b. Contraction Coefficient.—For sharp-edged circular orifices the coefficient of contraction ranges from about 0.61 to 0.72. It decreases with increasing size of orifice or with increasing head, which is just the opposite of the way in which the velocity coefficient varies. For a very small orifice the effects of the related phenomena of adhesion, surface tension, and capillarity reduce the contraction, thus giving large coefficients. As the size of the orifice increases, these factors become of decreasing importance and apparently cease to play any part for diameters of 2.5 in. or higher. That is, above this size the contraction is constant.

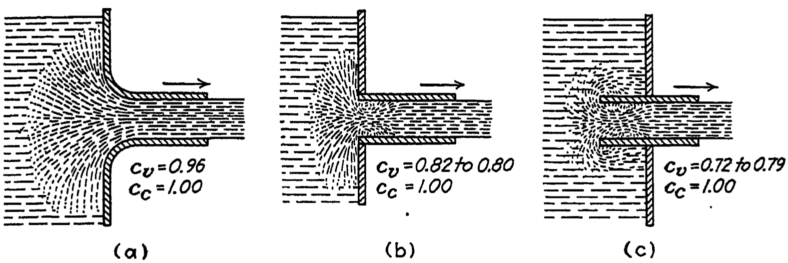


FIG. 91.—Coefficients for tubes.

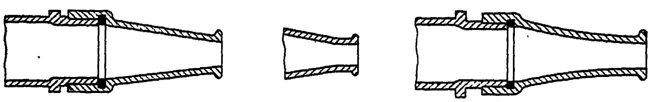
For a 2.5-in. orifice or larger the contraction coefficient under a head of 0.2 ft. appears to be about 0.622; at 1.5 ft. it is 0.615; and at 100 ft. or more it is constant at about 0.612. For a 1-in. orifice the coefficient varies from 0.676 to 0.638 as the head increases from 0.2 to 100 ft. For a ¾-in. orifice under a head of 0.2 ft., c_c appears to lie between 0.68 and 0.72; and for an infinitely small orifice the contraction coefficient approaches unity. However, few data have been obtained on velocity and contraction coefficients separately, so that it is impossible to state exactly how their values vary for small orifices and low heads. But it is clear that the change is slight above a certain head and that this *critical* head is higher as the orifice becomes smaller.

The preceding discussion applies to orifices for which there is complete contraction. This requires that the walls of the vessel be a distance away at least three times the diameter of a circular orifice or at least three times the smaller dimension for a rectangu-

lar orifice. If the distance is any less than this, the streamlines will be interfered with, and the contraction will be less, thus increasing the coefficient. As the wall becomes nearer, the contraction diminishes until finally it ceases altogether on that side where the wall and the orifice edge coincide. Unless the contraction is complete, the value of the coefficient is uncertain.

The contraction coefficient for various devices runs from about 0.52 up to 1.00 and thus varies through a wider range than is usual for the velocity coefficient. It is also much more sensitive to small changes in the edge of the orifice or the upstream face of the plate.

For nozzles with straight tips or so formed as to discharge a jet with strictly parallel streamlines, as in (*b*) and (*c*) of Fig. 92, there will be no contraction. In the case of the conical nozzle, such as in (*a*) of Fig. 92, the contraction increases with an increase in the vertex angle of the cone. The coefficient may be as low



(a)-Conical Nozzle (b)-Straight Tip (c)-Convex Nozzle

FIG. 92.—Standard nozzles.

as 0.80 for a vertex angle of 90 deg., but this is very extreme, and for normal types of construction the contraction coefficient for nozzles would usually be from 0.95 to 1.00.

For the purpose of measuring a fluid, one should select a standard orifice or nozzle for which coefficients may be assumed with some assurance, if the device is not to be calibrated. For this purpose a sharp-edged orifice is more reliable than some other types, but it does give a small size of jet. If the object is merely to discharge fluid, a type that has very little or no contraction appears to be better, as the increase in its contraction coefficient usually more than offsets any small decrease in its velocity coefficient. It should be emphasized that the value of the velocity coefficient is determined by the amount of friction loss, but the value of the contraction coefficient is not affected by anything except the nature of the streamlines. It merely determines how large a jet may flow from a given-sized orifice.

c. Discharge Coefficient.—The discharge coefficient is the one that is of the most practical utility, and it is also the one that

can most readily be determined experimentally. A vast number of experiments have been performed upon orifices and similar devices, but the values obtained by different observers do not always agree, especially for small orifices and low heads. These differences are probably due to slight differences in finish of edges and upstream faces of the plates and also to differences in the temperature, and consequently the viscosity, of the water. Since the flow of a free jet through an orifice involves both viscosity and gravity, it is seen that we are here concerned with both Reynolds number and Froude's number. Since the requirements of both of these cannot be satisfied at the same time, it is difficult to coordinate all the experimental data and deduce a general law. For orifices above 1.5 in. in diameter and for heads above 10 ft., the differences are not serious, but smaller orifices should be calibrated if they are to be used for measuring purposes.

Judd and King of Ohio State University found that for sharp-edged circular orifices the coefficient was practically constant for heads from 4 to 93 ft. An extract from their account in *Engineering News*, Sept. 27, 1906, is shown in Table V.

TABLE V.—COEFFICIENTS OF DISCHARGE

Orifice Diameter, Inches	Values of c
0.75	0.611
1.00	0.610
1.50	0.609
2.00	0.608
2.50	0.596

Table VI represents the work of H. J. Bilton of Australia, published in *Engineer*, June 19, 1908. This shows that the discharge coefficient attains a minimum value for each size at some "critical head" above which the coefficient remains unchanged. Hence the last values shown in the table may be used for all higher heads. Also, for any orifice 2.5 in. in diameter or larger and for any head of more than 17 in. the coefficient of discharge is 0.598. This is not identical with the minimum value given in Table V, but it merely illustrates the fact that an accuracy to three significant figures should not be expected in the case of standard orifices.

For square orifices the coefficients are very slightly larger than for circular orifices, and for rectangular orifices they are slightly

higher than for square ones, since the contraction must be relatively less.

TABLE VI.—COEFFICIENTS OF DISCHARGE

Head, in.	Diameter of orifice, in.						
	0.25	0.50	0.75	1	1.5	2	2.5
3	0.680	0.657	0.646	0.640			
6	0.669	0.643	0.632	0.626	0.618	0.612	0.610
9	0.660	0.637	0.623	0.619	0.612	0.606	0.604
12	0.653	0.630	0.618	0.612	0.606	0.601	0.600
17	0.645	0.625	0.614	0.608	0.603	0.599	0.598
18	0.643	0.623	0.613				
22	0.638	0.621					
45	0.628						

The discharge coefficients for tubes and nozzles may be estimated from the values of c_v and c_c that have been given. For straight tubes with sharp corners at entrance, the velocity coefficient is the determining item, while with nozzles the shape of the tip, which determines the contraction, is the item that introduces the principal variation. For a nozzle with no contraction a good average value is $c_d = 0.98$.

70. Nozzle Efficiency.—Since the jet from a nozzle is frequently employed for power purposes in a Pelton wheel, for example, we are interested in its energy efficiency. The efficiency of a nozzle may be defined as the ratio of the power in the jet to the power in the pipe at the base of the nozzle. Since the rate of discharge must be the same at the two points, the powers are proportional to the heads.

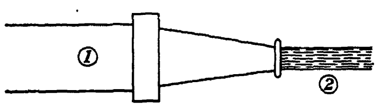


FIG. 93.

Thus, with reference to Fig. 93, $e = H_2/H_1$. But, since $H_2 = V^2/2g$, and from either Eq. (55) or Eq. (59) it may be seen that $V = c_v\sqrt{2gH_1}$, it follows that $H_2 = c_v^2H_1$. Hence the nozzle efficiency is

$$e = c_v^2. \tag{62}$$

This, however, is true only if all particles in the jet possess the same velocity and hence the same kinetic energy. Actually, V

is the average velocity of the jet, and it has been shown in Art. 48 that the true kinetic energy is greater than the value obtained by using the square of the average velocity. The nozzle efficiency would then be represented more accurately by the expression

$$e = c_v^2 + \frac{N^2}{H_1}. \tag{63}$$

For a good nozzle with a typical velocity profile, the true efficiency is about 1 per cent more than the value given by Eq. (62).

For a nozzle the relation between coefficient of loss and coefficient of velocity is shown by Eq. (57); that is, in Fig. 93,

$$h_f = \left(\frac{1}{c_v^2} - 1\right)\frac{V_2^2}{2g}.$$

The height to which a good fire stream can be thrown by a nozzle is from about two-thirds to three-fourths of the effective head at the base of the nozzle. The proportion is higher for large jets than for small ones, smooth nozzles than rough ones, and for low than for high pressures. For various information on nozzles see the references below.[1]

71. Square-edged Straight Tubes.—For a tube with a rounded entrance, as (*a*) in Fig. 91, there is only one condition of flow, which is that the tube shall flow full, as shown. But for a tube with a square entrance, as (*b*), there are two types of flow and two sets of coefficients. One is the case shown in Fig. 91, and the other is where the stream does not fill the tube but issues as a free jet from a square-edged standard orifice. In this case the coefficients are those whose values are given in Art. 69.

Even when the fluid fills the tube, the stream at *entrance* is a submerged jet with the same streamlines and coefficients as for a free jet through the same orifice. Owing to the smaller sectional area of this submerged jet, the velocity will be higher and the pressure lower than those of the stream at the discharge end of the tube. Let the velocity at the vena contracta be V_1, the area A_1,

[1] FREEMAN, JOHN R., *Trans. A. S. C. E.*, vol. 21, p. 303, 1889; *Trans. A. S. C. E.*, vol. 24, p. 492, 1891.

ECKART, W. R., JR., *Inst. Mech. Eng.*, Jan. 7, 1910.

FLEMING, V. R., *Proc. 5th Meeting Ill. Water Supply Assoc.*, 1913.

DAUGHERTY, R. L., "Hydraulic Turbines," 3d ed., p. 114, McGraw-Hill Book Company, Inc., 1920.

and the orifice coefficients for the entrance c_v' and c_c', while the velocity at the discharge end of the tube is V and the area A. Since there is no contraction of the stream leaving the tube, $c_d = c_v$. We then have the following:

$$V = c_v\sqrt{2gh}, \qquad A_1 = c_c'A, \qquad V = c_c'V_1,$$
$$q = A_1V_1 = AV = c_dA\sqrt{2gh},$$

while the friction loss between the reservoir and the vena contracta is $h_f = \left[\left(\frac{1}{c_v'}\right)^2 - 1\right]\frac{V_1^2}{2g}$, from Eq. (57). Writing the energy equation between those two points, and assuming the tube discharges into the atmosphere whose pressure is p_a, the *absolute* pressure at the vena contracta within the tube is

c_d

$w \quad w \quad | \backslash c_v'c$

where c' is the discharge coefficient of an orifice that is equivalent to the tube entrance. Taking $c_d = 0.82$ from Fig. 91 and $c' = 0.60$, as a typical value from either Table V or Table VI, this becomes $\frac{p_1}{w} = \frac{p_a}{w} - 0.865h$.

This equation shows that, as the head h increases, the pressure within the tube at the vena contracta decreases. When the head is raised to such a value that the pressure approaches the vapor pressure of the liquid, the stream will suddenly spring clear of the tube and issue through the orifice at its entrance as a free jet. The value of the head at which this transition occurs depends upon the barometer and vapor pressures, the amount of dissolved air in the liquid, and the coefficients c_d and c', but for water at normal temperature it usually lies between 38 and 48 ft. After the free jet is once established, the head may be lowered to a very low value before the stream will fill the tube again.

The same analysis applies to the reentrant tube shown in Fig. 91 (c), the only difference being that the numerical values of the coefficients are different. Because of the greater curvature of the streamlines at entrance, the contraction is greater for the free jet and also for the submerged jet at entrance when the tube is flowing full. The greater reexpansion causes more friction loss, and hence c_v for the full tube is less than for the previous tube.

Figure 94 shows a thin sharp-edged reentrant tube when it is not flowing full. In (*b*) of Fig. 91, the flow along the wall causes the pressure on it in the neighborhood of the orifice to be less than the static pressure, but the exact pressure values are not apparent. However, for the reentrant tube the liquid is practically at rest all over the wall, and hence the value of the pressure is known at every point on the surface. The only unbalanced pressure is that on an area opposite to the tube, and its value is whA_0, where A_0 is the area of the tube. The time rate of change of momentum due to the flow out of the tube is $(W/g)V$; but $W = wAV$, where A is the area of the jet, and $V^2/2g = c_v{}^2h$. Equating force to time rate of change of momentum,

$$whA_0 = wAV^2/g = wAc_v{}^2 2gh/g.$$

From this it is seen that

$$c_c = \frac{A}{A_0} = \frac{0.5}{c_v{}^2}. \qquad (65)$$

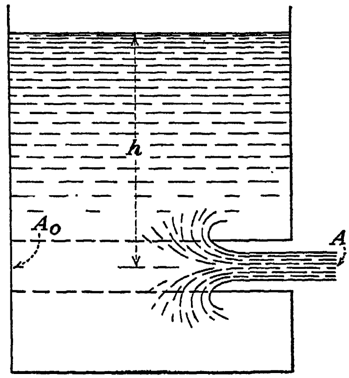


FIG. 94.

Assuming $c_v = 0.98$ for a typical case, $c_c = 0.52$. This type of tube seems to give the maximum contraction possible for a straight tube, and it is one for which the coefficient can be computed from pure theory.

Neither of these tubes is desirable for measuring purposes, but their theory has practical applications, especially to determining the entrance losses to long pipe lines. The values of the coefficients shown in Fig. 91 are subject to deviation for different diameters of tubes and for different ratios of length to diameter.

A very short tube, whose length is much less than its diameter, may discharge a free jet which does not fill the tube but may have its flow transformed to the other type where the stream fills the tube, if it operates submerged. Hence, for such cases the coefficient of discharge may be higher when submerged than when it is not.

72. Diverging Tube.—In Fig. 95, since there is no contraction of the stream leaving the tube, $c_d = c_v$; and at the end of the tube, where the area is A, we have $V = c_v\sqrt{2gh}$ and $q = c_d A\sqrt{2gh}$,

just as in the case of other tubes. The principal difference is that the value of c_d varies with the angle of divergence as well as with the length of the tube. The coefficient decreases quite rapidly with increasing angles of divergence.

By the equation of continuity, it is seen that the velocity at the *throat* of the tube is higher than it is in the jet discharging from the tube and that consequently the pressure there is much less than that in the jet. If the entrance is square edged, there will be a contraction of the stream at the throat, which will accentuate the difference still more. If A_0 is the tube area at the throat, while c' is the discharge coefficient for an orifice equivalent to the entrance to the tube up to the throat, then by a procedure identical with that in the preceding article it may be shown that the *absolute* pressure at the throat is

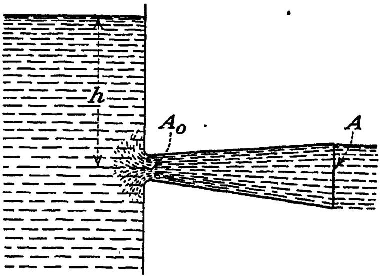


Fig. 95.

$$\frac{p_0}{w} = \frac{p_a}{w} - \left[\left(\frac{c_d}{c'}\right)^2\left(\frac{A}{A_0}\right)^2 - 1\right]h. \tag{66}$$

It is seen that Eq. (64) is really a special case of this, where $A_0 = A$. And on account of the influence of the ratio of the areas, there will always be two types of flow possible here whether the tube has a square-edged or a rounded entrance; also, the critical head at which the transition occurs will be lower.

When the head h is such that a free jet issues from the entrance without touching the sides of the tube, the case is like that of any other orifice. But when the conditions are such that the stream fills the tube, it is seen that the rate of discharge is very much greater than that obtained from any orifice whose diameter is that of the throat. The velocity at the throat is really $V_0 = c_v'\sqrt{2g\left(h + \frac{p_a}{w} - \frac{p_0}{w}\right)} = c_v'\sqrt{2gh\left(\frac{c_d}{c'}\right)^2\left(\frac{A}{A_0}\right)^2}$, which is obviously higher than the value with a plain orifice. Although it is quite illogical, the rate of discharge can be expressed as the area of the throat times the ideal velocity of the jet from the end

of the tube multiplied by a coefficient whose value is such as to offset this incorrect procedure. That is,

$$q = A_0V_0 = AV = c_0A_0\sqrt{2gh}.$$

From this it is seen that $c_0 = c_dA/A_0$. As the angle of divergence of the tube increases or as its length increases, the value of c_d will decrease, but it will not decrease so rapidly as the ratio A/A_0 increases. Consequently, c_0 is always greater than unity, and values up to 2.5 and even up to 5 have been reported. The only merit of this hybrid coefficient is that it indicates directly the relative discharge capacity of a diverging tube relative to an orifice of the same size as the throat.

It is obvious that there are limits to the angle of divergence for which the tube will function in the manner assumed. Even if the tube is submerged, the stream will not fill it if the angle is too great but will flow as a submerged jet with eddies filling the space between the jet and the tube walls. Diverging tubes used in practice are usually limited to cone vertex angles of less than 10 deg., where efficient operation is desired.

73. Vertical Orifice under Low Head.—In the case of an orifice whose vertical dimensions are large as compared with its depth below the free surface, it is necessary to proceed as follows: Choose an elementary area dA such that all portions are at the same depth z below the free surface. Now by Art. 66 the rate of discharge through this strip may be expressed as

$$dq = c_d\sqrt{2gz}\,dA.$$

The rate of discharge through the entire orifice may be obtained by integrating the preceding. Thus

$$q = c_d\sqrt{2g}\int z^{1/2}\,dA. \tag{67}$$

In the case of a rectangular orifice (Fig. 96) take $dA = b\,dz$, and after integrating obtain

$$q = c_d\tfrac{2}{3}\sqrt{2g}\,b(h_2^{3/2} - h_1^{3/2}). \tag{68}$$

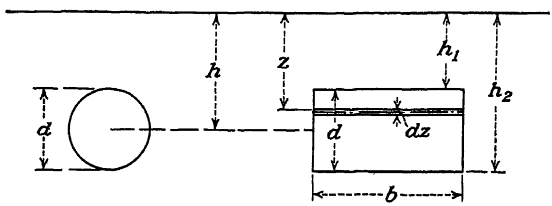
If h = depth of center of gravity below the free surface,

$$h_2 = h + \frac{d}{2}; \qquad h_1 = h - \frac{d}{2}.$$

Expanding $\left(h + \frac{d}{2}\right)^{3/2}$ and $\left(h - \frac{d}{2}\right)^{3/2}$ by the binomial theorem and substituting in Eq. (68), the following is obtained:

$$q = c_d bd\sqrt{2gh}\left(1 - \frac{d^2}{96h^2} - \frac{d^4}{2048h^4} - \cdots\right). \tag{69}$$

The expression in parentheses is a rapidly converging series, and its value is always less than unity. When $h = d$ the value of this factor is 0.989, while for $h = 2d$ its value becomes 0.997. Thus for any head greater than $2d$ the rate of discharge may be obtained by the simpler formula $q = c_d A\sqrt{2gh}$.



(a)-Circular Orifice (b)-Rectangular Orifice

FIG. 96.

In similar manner the rate of discharge through a circular orifice of area A is given by the expression

$$q = c_d A\sqrt{2gh}\left(1 - \frac{d^2}{128h^2} - \frac{5d^4}{16{,}384h^4} - \cdots\right). \tag{70}$$

It may be found that when $h = 2d$ the value of this series is 0.998, thus indicating that the use of the exact formula is unnecessary for heads above that value.

The nozzle is a device of considerable practical importance; but orifices in tank walls and short tubes are seldom used. While the information given about them may occasionally be useful, the principal value in the study of their theory and their coefficients is in enabling the student to learn the methods of applying the fundamental principles of the preceding chapter. Also, this study is an aid in understanding more complex problems which arise later. The remainder of this chapter is devoted to those devices which are commonly used in the determination of fluid flow.

EXAMPLES

145. In Fig. 84 the actual measured diameter of the minimum section of the jet was $6^{13}/_{16}$ in., the area of the nozzle opening being 43.02 sq. in. Compute the coefficients of velocity, contraction, and discharge, using the values of H_1 and V_2 given. What is the efficiency of the nozzle? What is the horsepower in the jet?

Ans. 0.989, 0.846, 0.837, 97.8 per cent, 5,250 hp.

146. What is the value of the head lost in hydraulic friction in the nozzle of Fig. 84? What is the value of k? *Ans.* 17 ft., 0.022.

147. Find values of the coefficient of loss when the velocity coefficient has values of 1.00, 0.99, 0.95 and 0.80. Find c_v when k has values of 0, 0.5, 1.00.

Ans. $k = 0.562$, $c_v = 0.707$ for last values in each set.

148. Compute rate of discharge for a 2-in. standard circular orifice under a head of 100 ft., using both the Judd-King and the Bilton coefficients.

149. Compute the rate of discharge through a standard circular orifice 0.25 in. in diameter under a head of 12 in.

150. A diverging tube has a throat diameter of 1 in. and a diameter of 2 in. at the discharge end. If $c' = 0.95$, $c_d = 0.55$, barometer = 33 ft. of water, and the minimum allowable pressure is 1 ft. of water, what is the maximum possible value of h? *Ans.* 7.39 ft.

151. What is the ratio of the rate of discharge through the tube in Prob. 150 to the value if the tube were to be cut off at the throat; to that of a standard circular orifice of the same size as the throat? *Ans.* 2.32, 3.66.

152. What is the velocity of the flow through the throat of the tube in Prob. 150?

74. Weir.—A weir is a special form of orifice, its distinguishing feature being that it is placed at the water surface so that the head on its upper edge is zero. Thus the usual formulas for orifices given in Art. 66 can no longer be applied, and the methods of Art. 73 must be employed. The weir is one of the most widely accepted standard devices for the measurement of water.

If it be assumed that the velocity of water through an orifice varies as the square root of the depth, the curve in Fig. 97 would give a true representation of the flow. However, the particles of water at the surface of the weir opening do not remain at rest but flow with considerable velocity. It may also be observed that the level of the water at this point drops below its normal value, as shown in Fig. 98. Also, it must be noted that the streamlines flowing through the weir are not necessarily normal to the plane of the weir; hence, it is hardly correct to multiply their velocities by areas in the plane of the opening. For these and other reasons it is impossible to derive by theory weir formulas that are exactly correct, but they serve as expressions that may be made to yield

correct results by the proper choice and use of experimental coefficients.

It might seem natural to measure the depth of water flowing over the crest of a weir, but in practice it is difficult to do this with any degree of accuracy. It is found more feasible to measure the elevation, above the weir crest, of the water surface at some distance back from the weir, where the water is relatively

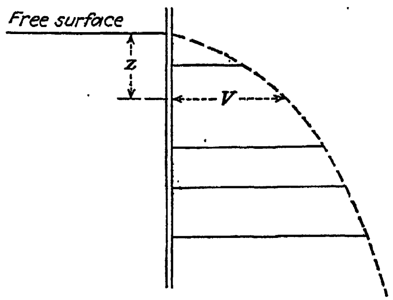


FIG 97.

quiet. Thus, all weir formulas express the rate of discharge as a function of H (Fig. 98). This measurement must be taken at a point far enough back to avoid the effects of the surface curve. This distance should be at least $2.5H$ but preferably $4H$ or more, yet not so far away that the normal slope of the water surface affects its readings to any appreciable extent. The usual instru-

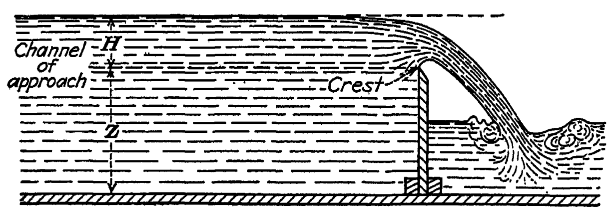


FIG. 98.—Weir.

ment for measuring H is the hook gage, one form of which is shown in Fig. 99. The gage should be mounted on a rigid support. In using it the sharp-pointed hook is submerged beneath the surface and then carefully raised until a slight distortion may be seen on the water surface. The hook should then be lowered until this distortion barely disappears. From this reading the value of H is obtained by subtracting the "hook-gage zero,"

which is the reading of the gage when its point is just level with the crest of the weir, as the lower edge is called.

The drop in the surface as the water flows over the weir is called *surface contraction*, while the rise of the bottom part of the stream above the crest, as shown in Fig. 98, is called *crest contraction.* The area of the stream at a section where the streamlines become substantially parallel is less than the area BH, where B is the length of the weir crest. Thus the contraction of the stream from a weir is similar to that of a jet from an orifice. Some types of weirs, as will be seen later, also have side contractions so that the width of the stream issuing through the opening is less than the distance B.

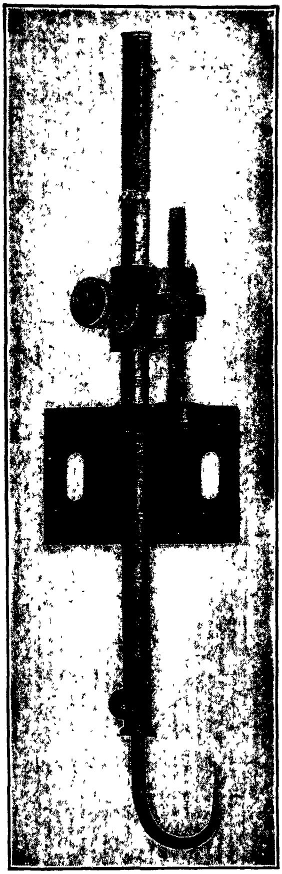


FIG. 99.—Hook gage. (*Courtesy of W. & L. E. Gurley.*)

In order that the crest contraction shall be complete, it is necessary that the bottom of the channel be a sufficient distance below the crest of the weir and that the weir crest be sharp or at least square edged, and the plate not too thick. The higher the head on the weir, the greater is the thickness of the plate for which the stream will clear the other edge and thus behave the same as if the crest were a knife-edge.

The Suppressed Rectangular Weir. Probably the most common type of weir is one whose shape is rectangular. If the width of the weir is the same as that of the channel of approach, as in Figs. 100 and 101, the stream of water flowing over the crest will not undergo any lateral contraction; that is, the end contractions are suppressed. With this type of weir it is customary to extend the sides of the channel beyond the crest so that the falling stream is bounded by them. If these si des are not so extended, the stream will expand somewhat, and the discharge for a given value of H will be slightly larger than in the standard type.

It is necessary in this or any type of weir to insure that the weir is "ventilated," that is, that air has access to the underside

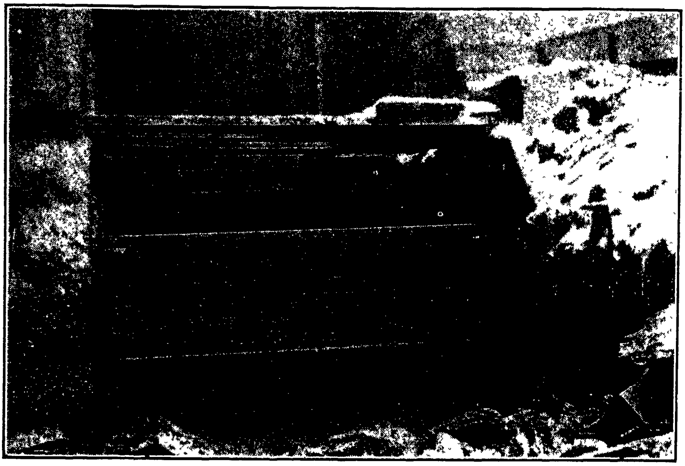


FIG. 100.—Rectangular weir without end contractions.

of the falling water. Otherwise, the air will be gradually swept out and the water will tend to cling to the face of the weir instead

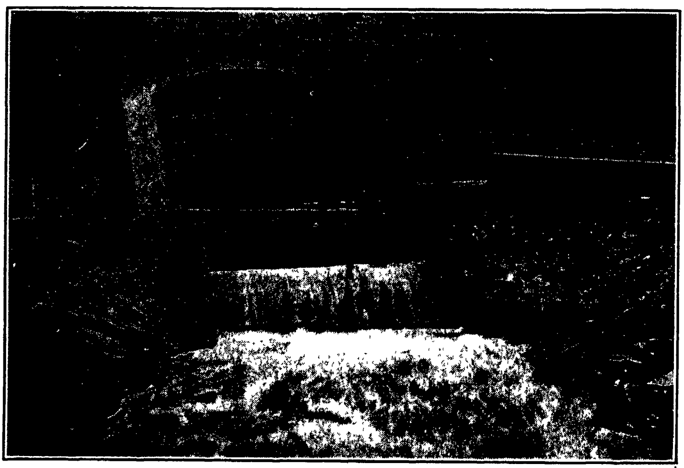


FIG. 101.—Rectangular weir without end contractions.

of springing clear of it. For a given value of H the rate of discharge would then be greatly increased and the usual coefficients would no longer apply.

Contracted Rectangular Weir.—When the width of the weir is less than that of the channel of approach, as in Figs. 102 and 103, the contraction that occurs at each end causes the real width of

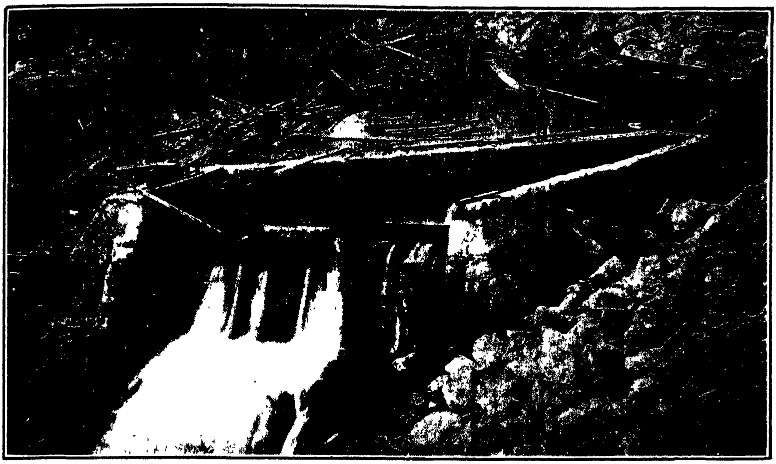


FIG. 102.—Rectangular weir with end contractions. (*From a photograph by F. H. Fowler.*)

the stream of water to be less than that of the weir itself. Such a weir is called a *contracted weir.*

Trapezoidal or Cippoletti Weir.—The trapezoidal weir is one in which the sides of the notch are not vertical but diverge so

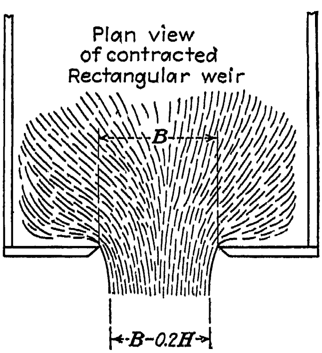


FIG. 103.

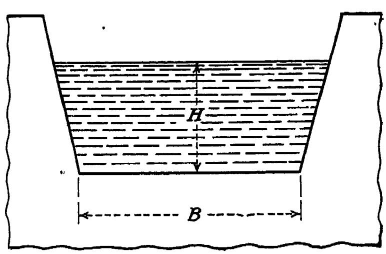


FIG. 104.—Trapezoidal weir.

that the width at the water surface increases. If the side slopes have the ratio of 1:4, the weir is called a *Cippoletti* after an Italian engineer of that name who proposed it.

The Triangular Weir.—The triangular weir, such as is shown in Fig. 105, is useful for measuring relatively small rates of discharge, as a reasonable value of H may be obtained by employing a sufficiently small vertex angle. But for discharges much above

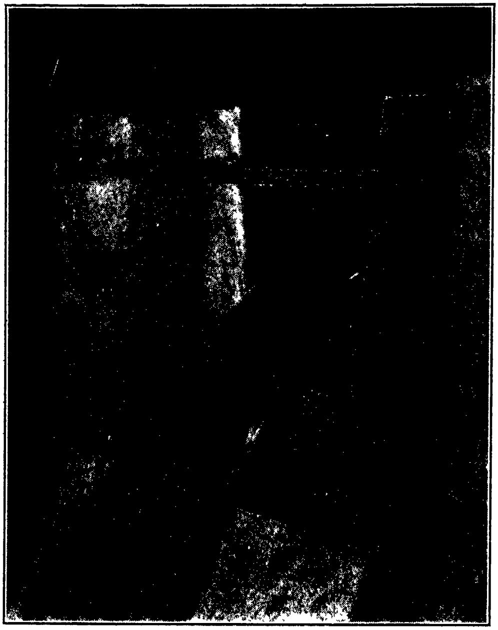


FIG. 105.—Discharge from a 60-deg. triangular weir.

2 or 3 cu. ft. per sec. excessively high values of H are necessary, and other types of weir would then be used.

75. The Rectangular Weir.—For the rectangular weir in Fig. 106, the discharge through the elementary strip of area dA is given by

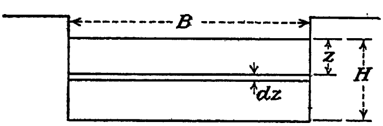


FIG. 106.

$$dq = \sqrt{2gz}\, B\, dz. \tag{71}$$

Integrating between limits and introducing a coefficient of discharge c_d,

$$q = c_d\sqrt{2g}B\int_0^H z^{1/2}\, dz,$$

$$q = c_d(2/3)\sqrt{2g}BH^{3/2}. \tag{72}$$

This equation is the basis of all rectangular weir formulas, which usually differ from each other only in their manner of expressing the value of the coefficient of discharge and in correcting for the velocity in the channel. This coefficient is not a constant but varies with H, Z, velocity of approach, sharpness of crest, smoothness of upstream face of weir plate, and the viscosity, density, and surface tension of the water. These last three quantities are primarily dependent upon the temperature.

However, the exact way in which the coefficient varies with each one of the foregoing factors is not known. The fact that experimental results obtained by different observers do not agree is merely because of the differences in some or all of the preceding quantities and because of the fact that the data have not been coordinated by the similarity laws, which become difficult of application when so many variables are involved. Hence, the exercise of judgment based on experience is necessary in the selection of a numerical coefficient or the formula to use in a specific case.

If the velocity of approach v_0 at the section where H is measured is appreciable, the limits of integration should then be $H + v_0^2/2g$ and $v_0^2/2g$, and the resulting fundamental equation is

$$q = c_d \frac{2}{3}\sqrt{2g}B\left[\left(H + \frac{v_0^2}{2g}\right)^{3/2} - \left(\frac{v_0^2}{2g}\right)^{3/2}\right] \qquad (73)$$

If the velocity of approach were uniform, this equation would be correct, but in any normal case the velocity near the surface of the stream is higher than the average, and it has been found that the flow over the weir is much more affected by the velocity in that portion of the stream above the level of the weir crest than it is by that below it. Hence, if $h_0 = v_0^2/2g$ is the velocity head due to the *average* velocity, the velocity head in the portion of the stream above the level of the crest may be denoted by αh_0 where α ranges in value from 1 to 2.2 or more. The former value would apply if the velocity were uniform, and the latter, or an even higher value, where the surface velocity was much higher than the average velocity. As a rough rule, α is sometimes taken as the ratio of the surface velocity to the average velocity. Schoder recommends that when $H = Z$ or is even more than Z, the value of α should be taken as 1.0 to 1.3 but that the value should increase up to 2.0 or more as the ratio H/Z decreases. He

states that for real precision an exact knowledge of the velocity distribution in the channel is essential.[1]

As the ratio H/Z decreases, the importance of h_0 decreases; and when H/Z is $\frac{1}{5}$ or less, the error due to the neglect of the velocity of approach is less than other uncertainties and errors. In such a case Eq. (72) may be employed.

Substituting the term αh_0 in Eq. (73), we have

$$q = c_d \frac{2}{3}\sqrt{2g}B[(H + \alpha h_0)^{3/2} - (\alpha h_0)^{3/2}] \tag{74}$$

$$= KB[(H + \alpha h_0)^{3/2} - (\alpha h_0)^{3/2}]. \tag{75}$$

The equation in this form would have to be solved by trial. To do so it is necessary first to solve by Eq. (72) in order to obtain an approximate value of q. This value is then divided by the channel area at the section where the hook gage is located in order to find v_0 and a corresponding value of αh_0. This value is inserted in Eq. (74) or (75), and a value of q determined which will be somewhat larger than the preceding. This new and larger value of q is used to find a new v_0, and the process repeated. However, after about two such solutions have been made, it will be found that any further solution will increase the result only a negligible amount.

In order to avoid a trial solution, the same steps can be taken in the algebraic formulas, and a form evolved in which trial will not be necessary in a numerical case. Thus Eq. (75) may be written in the equivalent form

$$q = KBH^{3/2}\left[\left(1 + \frac{\alpha h_0}{H}\right)^{3/2} - \left(\frac{\alpha h_0}{H}\right)^{3/2}\right],$$

where the expression in parentheses is the correction for the velocity of approach. This is then expanded by the binomial theorem, and all terms after the second one are dropped because (a) their sum is less than $(\alpha h_0/H)^{3/2}$, which in itself is a small quantity; and (b) they are all positive and hence nearly cancel $-(\alpha h_0/H)^{3/2}$. The result is

$$q = KBH^{3/2}\left[1 + \frac{3}{2}\frac{\alpha h_0}{H}\right]. \tag{76}$$

[1] SCHODER, E. W., section on hydraulics in "Marks's Mechanical Engineers' Handbook," McGraw-Hill Book Company, Inc., 1930.

From this equation, $v_0 = \dfrac{q}{A} = \dfrac{KBH^{3/2}\left(1 + 1.5\alpha\dfrac{h_0}{H}\right)}{B(H + Z)}$, and therefore, $h_0 = \dfrac{v_0{}^2}{2g} = \left(\dfrac{K^2}{2g}\right)\cdot\dfrac{H^3\left(1 + 1.5\alpha\dfrac{h_0}{H}\right)^2}{(H + Z)^2}$. Substituting this value of h_0 in Eq. (76), we have

$$q = KBH^{3/2}\left[1 + \frac{3}{2}\alpha\frac{K^2}{2g}\left(\frac{H}{H + Z}\right)^2\left(1 + 1.5\alpha\frac{h_0}{H}\right)^2\right].$$

This expression still involves h_0, which is the unknown that we wish to eliminate, but it is here a secondary part of a term, which in turn modifies a secondary part of the main equation. Hence, some inaccuracy in its value here makes very little difference in the final result. In order to evaluate $\left(1 + 1.5\alpha\dfrac{h_0}{H}\right)^2$, note that it is the same expression as the term in brackets in Eq. (76), except that it is squared. Letting this correction factor be represented by y, we then have that approximately

$$y = \left(1 + 1.5\frac{K^2}{2g}\alpha\left[\frac{H}{(H + Z)}\right]^2\right)^2.$$

For an extreme case where the velocity of approach is negligible, that is, H/Z is very small, $y = 1$. If $H = Z$ and $\alpha = 1.15$, and K be taken as 3.33, which it is very nearly, then $y = 1.157$. If $\alpha = 1.5$ while $H = 2Z$, $y = 1.375$. Since the ratio of H/Z is rarely that large, and since the value of α for such a case would probably be less than 1.5, since α decreases as H/Z increases, it is seen that this is a very extreme case. Thus the equation might be written.

$$q = KBH^{3/2}\left[1 + 0.259y\alpha\left(\frac{H}{H + Z}\right)^2\right], \tag{77}$$

where, in the opinion of the author, y lies between the limits of 1 and 1.375. In fact, for most normal cases it is probably between 1.04 and 1.20. But the uncertainty in the value of α is much greater than the probable variation in y. However, in a specific case where values of H/Z and α are given, the value of y may be computed in the manner shown. But the error in the final result

will be small if a value of y is merely assumed, bearing in mind that y tends to increase as H/Z increases.

Equation (77) is not only useful in avoiding the necessity of solving Eq. (75) by trial, but it is also a fundamental form upon which certain special weir formulas, such as that of Bazin, are based.

76. Formulas for Weirs without End Contractions.—For rectangular weirs whose width B is the same as that of the channel of approach so that end contractions are suppressed, the following are some of the more commonly used formulas. Many other equations may be found in the literature of the subject, but the ones given here are representative.

a. Francis Weir Formula.—In 1848–1852 at Lowell, Mass., James B. Francis made some precise investigations of the flow of water over weirs of large size.[1] His observations showed that the flow varied as $H^{1.47}$, but he adopted $H^{1.5}$ for convenience in computation, since $H^{3/2} = H\sqrt{H}$. If the coefficient were a constant for the former, it is obvious that it must vary for the latter and that it must decrease slightly as the head increases. However, from his observations he selected a constant value of 0.622 for c_d in Eq. (72) so that for a weir with negligible velocity of approach

$$q = 3.33BH^{3/2}; \tag{78}$$

and for a weir with velocity of approach considered he gave

$$q = 3.33B[(H + h_0)^{3/2} - (h_0)^{3/2}]. \tag{79}$$

For a negligible velocity of approach or for a uniform velocity, Schoder and Turner at Cornell University have found the Francis formula to be about 7 per cent too low when $H = 0.1$ ft., about 3 per cent too low when $H = 0.2$ ft., but to be accurate within 1 to 3 per cent for heads above 0.3 ft. Schoder and Turner have verified this for heads from 0.011 to 2.75 ft., while in 1899 G. W. Rafter and others conducted experiments at Cornell University with values of H up to 4.68 ft. and found the formula to hold up to that head.[2]

[1] FRANCIS, JAMES B., "Lowell Hydraulic Experiments," Van Nostrand, 1868.

[2] SCHODER, E. W., and K. B. TURNER, "Precise Weir Measurements," *Trans. A.S.C.E.*, vol. 93, p. 999, 1929.

RAFTER, G. W., "On Flow of Water over Dams," *Trans. A.S.C.E.*, vol. 44, p. 220, 1900.

In the Francis experiments the maximum value of H/Z for suppressed weirs was 0.218, and the velocity of approach was consequently low, being only 0.559 ft. per sec. For higher velocities of approach and for unequal velocity distribution it is better to use the modified Francis formula

$$q = 3.33BH^{3/2}\left(1 + 1.5\alpha\frac{h_0}{H}\right) \tag{80}$$

or the equivalent form

$$q = 3.33BH^{3/2}\left[1 + 0.259y\alpha\left(\frac{H}{H + Z}\right)^2\right]. \tag{81}$$

where y varies from about 1.04 to about 1.20, and α from 1 to 2.

b. Fteley and Stearns Formula.—In 1877–1879 the authors of this formula conducted experiments in connection with the Sudbury aqueduct near Boston and like Francis used weirs of relatively large size. From their own tests and from a study of the Francis data they proposed

$$q = 3.31B(H + 1.5h_0)^{3/2} + 0.007B. \tag{82}$$

Although they found α to vary from 1.44 to 1.87, they used a constant value of 1.5 in the equation. The term 0.007 is not an abstract number but must obviously have the dimensions of square foot per second. This last term is doubtless due to the surface tension of the water which makes it cling to the crest at low heads. It is seen that this constant term in the equation becomes of increasing importance as the head diminishes and that thus it tends to overcome the error that is found in the Francis formula for very low heads. Hence this equation is often used for heads of from 0.07 to 0.5 ft. in preference to the Francis formula.

c. Bazin Formula.—This formula is based upon extensive investigations of H. Bazin in France which were begun in 1886 and carried on for a number of years. A particular merit of his work is that he had some very high velocities of approach due to very low heights of weir crests. The maximum value of H/Z was 1.7, and the velocity of approach was very nearly 3 ft. per sec. He thus covered a wider range of conditions than others.

Bazin's equation is of the form shown in Eq. (77). Although he found values of α as high as 2.21, Bazin assumed α to have a

constant value of $\frac{5}{3}$ and found that his experimental data were represented within about 1 per cent by the following:

$$q = \left[3.25 + \frac{0.0789}{H}\right]\left[1 + 0.55\left(\frac{H}{H + Z}\right)^2\right]BH^{3/2} \tag{83}$$

The first term in this equation is the value of K and is made to diminish with H, as it should. The second term is the correction for the velocity of approach. This formula is said to be quite accurate for heads from 0.1 to 0.3 ft. but to give results from 2 to 3 per cent too high for $H = 0.3$ to 1.2 ft. Being devised for a constant value of α, it cannot be applied with various velocity distributions in the channel. Its merit is that it is based upon a wide range of the ratio H/Z and is more flexible in that respect.[1]

d. Rehbock Formula.—Small-scale but very precise experiments covering a wide range of conditions have been carried on by T. Rehbock at Karlsruhe Hydraulic Laboratory, in Germany, since 1904. In 1912 this formula was published:

$$q = \left(3.237 + \frac{5.35}{320H - 3} + 0.428\frac{H}{Z}\right)BH^{3/2} \tag{84}$$

Rehbock has more recently simplified this to

$$q = \left(3.237 + 0.428\frac{H}{Z}\right)B(H + 0.00281)^{3/2}. \tag{85}$$

It gives values that are very similar to those obtained with either the Francis or the Fteley and Stearns formulas. It is based upon flow in a channel with a fairly uniform velocity and contains no provision for the effect of a variation in velocity distribution.

77. Formulas for Weirs with End Contractions.—When the width B of a rectangular weir is less than that of the channel, there will be a lateral contraction of the stream so that its width is less than that of the weir. Francis concluded that the effect of each side contraction was to reduce the width of the stream by $0.1H$. The usual contracted weir will have a contraction at each end, but occasionally a weir is placed against one side wall so

[1] Nagler, F. A., "Verification of the Bazin Weir Formula by Hydrochemical Gagings," *Trans. A.S.C.E.*, vol. 83, p. 105, 1919. This is an excellent discussion of the Bazin formula and a report of tests up to $H = 4$ ft Nagler's conclusion is that the Bazin formula is correct at all heads.

that the contraction on one end is suppressed. If n = the number of contractions, which may be 2, 1 or 0, the Francis formula is

$$q = 3.33(B - 0.1nH)H^{3/2}. \tag{86}$$

This correction is purely empirical and is not altogether logical, but no better method for it has yet been devised. It is seen that if H is only large enough relative to B, the formula would indicate zero flow and that for a larger H the flow would be negative. This emphasizes the fact that the formula is strictly limited in its range. Francis stated that B should be greater than $3H$. He also stated for perfect crest contraction the distance from the bottom should be at least $3H$. This distance may be reduced, however, to about $2H$ without increasing the rate of discharge more than about 1 per cent. Figure 107 shows the absolute

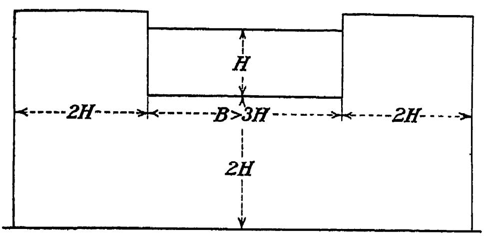


FIG. 107.—Limiting proportions of standard contracted weirs.

minimum dimensions that are allowable for such a weir, but larger values are desirable.

For the velocity of approach to be negligible, the area of the channel should be about six or more times the area of the weir BH. For a suppressed rectangular weir, that would mean the channel depth would have to be $6H$ or more. Such a depth is often impractical, and hence the correction for velocity of approach must usually be made. In the case of the contracted weir, with the proportions shown in Fig. 107, the area of the channel is seven times BH, the area of the weir opening. Since this is the minimum channel area permissible, it is seen that the correction for velocity of approach is hardly necessary for a standard weir with two contractions. However, if it is desired to make the correction, it can be done in the Francis formula by substituting this corrected dimension in Eq. (73) or (75). The same correction for B should also be made in the Fteley and

Stearns formula, but they recommended a value of 2.05 for α in that case.

For a weir with two end contractions Gourley and Crimp, on the basis of tests conducted by them, derived the formula

$$q = 3.10B^{1.02}H^{1.47}. \tag{87}$$

This seems to agree very well with Eq. (86) and has the merit that it apparently has no such limitations imposed upon the maximum value of H/B.

For a narrow rectangular notch where H equals or exceeds B, Schoder gives

$$q = 3.00BH^{3/2}. \tag{88}$$

Thus the entire possible range seems to be covered.

78. The Cippoletti Weir.—In order to avoid the trouble of correcting for end contractions, the sides of the Cippoletti weir are given such a batter (1:4) that they are supposed to add enough to the effective width of the stream to offset the contraction $0.2H$ of the contracted Francis weir. Thus computations may be made upon the basis of the width B at the crest by the following formula:

$$q = 3.367BH^{3/2}. \tag{89}$$

This weir is often used in irrigation work where precision is not important, but it is not considered so reliable a metering device as other forms.

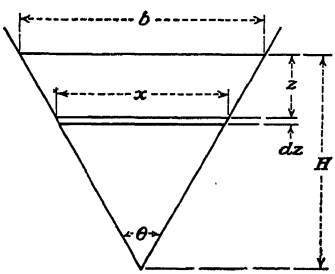


FIG. 108.

79. The Triangular Weir.—Figure 108 is a triangular weir with any vertex angle θ. The rate of discharge through an elementary strip of area dA will be

$$dq = c_d\sqrt{2gz}\,dA.$$

Now, $dA = x\,dz$; and by similar triangles $x:b = (H - z):H$. Hence, $dA = (b/H)(H - z)\,dz$. Substituting in the foregoing, the following is obtained for the entire notch:

$$q = c_d\sqrt{2g}\frac{b}{H}\int_0^H (H - z)z^{1/2}\,dz.$$

Integrating between limits,

$$q = c_d\sqrt{2g}\frac{b}{H}\left(\frac{2}{3}H^{5/2} - \frac{2}{5}H^{5/2}\right).$$

But $b = 2H \tan \theta/2$. Inserting this and reducing,

$$q = {}^8\!/_{15} c_d\sqrt{2g} \tan \frac{\theta}{2} H^{5/2}. \tag{90}$$

This expression for any given weir may be reduced to

$$q = KH^{5/2}.^{1} \tag{91}$$

In Figs. 109 and 110 may be found experimental values of K for several triangular weirs. The 54-, 60-, and one of the 90-deg.

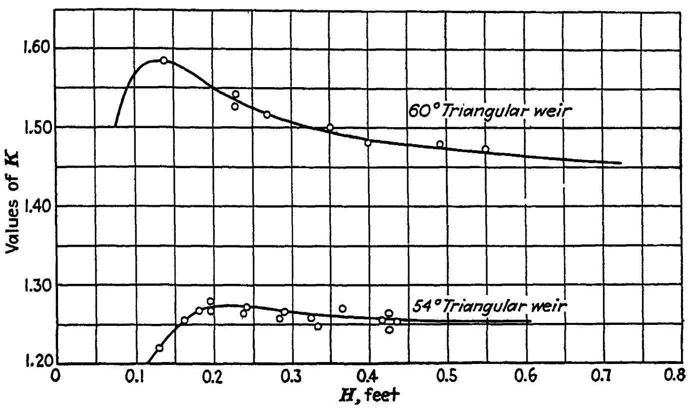


FIG. 109.—Coefficients of triangular weirs.

weirs are in the laboratory of Sibley School of Mechanical Engineering, Cornell University.[2] The two lower curves in Fig. 110 were plotted from data in a valuable paper by James Barr.[3] The weir for which the very lowest curve was constructed had a very fine sharp edge, while the other weir had a square corner and a thickness of about $\frac{1}{16}$ in. Both of these weirs have values of K below that of the Cornell University weir, but the difference, of about 1 per cent, may be due to the difference in smoothness of

[1] $(H^{5/2} = H^2\sqrt{H}.)$

[2] *Eng. News*, vol. 73, p. 636, 1915.

[3] "Experiments upon the Flow of Water over Triangular Notches," *Engineering*, Apr. 8 and 15, 1910.

the surface of the plate in which the notch is cut. A rougher surface has been found to decrease the contraction of the stream and thus increase the weir coefficient.

These curves show that the discharge does not vary as the five-halves power of H, since c_d is not a constant. Thompson, who first employed the triangular weir, chose for K a value of 2.54. This may be seen to be a fair average for ordinary weirs; but for greater precision, values of K may be read from the curves for the particular value of H. As a fair average for heads above 0.2 ft.,

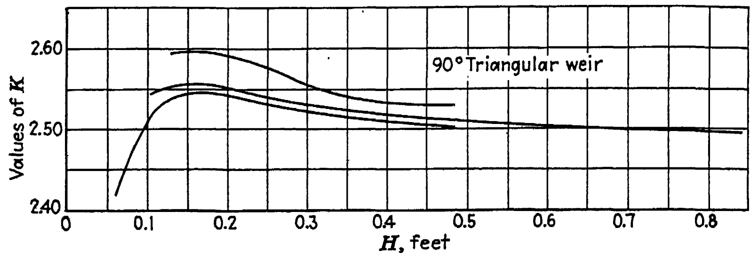


Fig. 110.—Coefficients of 90-deg. triangular weirs.

it may be found that the curves here shown will give for the 60-deg. weir

$$q = 1.42H^{2.44} \tag{92}$$

and for the 90-deg. weir

$$q = 2.48H^{2.475}. \tag{93}$$

80. Construction of Weirs.—In order that the standard formulas may be applied to a weir with some assurance of accuracy, certain details of construction and proportion must be observed. The weir plate should be set with its face vertical and the crest absolutely level and should be normal to the direction of flow in the channel. Adequate provision should be made for the admission of air to the space between the sheet of falling water and the downstream face of the weir. If no air is admitted, the air that may be trapped in the case of a suppressed weir will be completely swept out, and the water will then cling to the face of the weir. If the aeration is inadequate, there may be a partial vacuum, which will reduce the crest contraction. The crest should be sharp. For low heads a *knife-edge* is necessary. For moderate heads a thin plate with a perfectly square corner may be employed.

In the case of a suppressed weir the side walls should extend beyond the weir plate so as to confine the sides of the stream. If this is not done, the value of K should have $0.2H/B$ added to it.[1]

The head should be measured in a stilling box outside the weir channel but connected with it through a pipe leading from a small piezometer opening in the side of the channel. This opening should be in a place where the wall is smooth and where the stream flow is not disturbed. For accuracy it may be desirable to have a second connection on the opposite side of the channel. This opening should be a distance upstream from the weir plate at least $2.5H$ and preferably more. The breadth of the weir should be so chosen that the head to be measured is not too low. The head on the weir should preferably be above 0.2 ft. Values under that are difficult to measure with precision, since a small error in the reading becomes of large percentage value. Also, for low heads the variation in the crest contraction may be considerable, owing to roughness of the surface of the plate, the lack of sharpness of the weir crest, and the effect of surface tension which tends to cause the water to adhere to the top of the crest instead of springing clear. All of these factors decrease the crest contraction and thus increase the flow over the weir for the same head reading. The main trouble, however, is that the results are variable and uncertain. The effects of all of these factors are minimized under high heads.

Another reason for avoiding a very low head on a weir is that viscosity also produces more effect on the crest contraction than at higher heads. Harris found that for a head of 0.18 ft. a reduction of water temperature from 80 to 35°F., which would practically double the viscosity, increased the discharge about 2.4 per cent.[1]

The channel upstream from the weir should be straight and level and free of all disturbing influences for a sufficient distance to permit the stream to assume a normal quiet flow. Baffles, surface floats to kill waves, etc., should not be located too near the weir plate. The length of uniform channel that is desirable has not yet been established, but values for it have been variously given as 25 ft., $36H$, and from 6 to $10(H + Z)$.

[1] Harris, C. W., *Bull.* 81, Eng. Exp. Sta., Univ. Washington.

81. Errors in Weir Measurements.—Although an accuracy of a fraction of 1 per cent is sometimes claimed for a weir, such a result is improbable, except perhaps in a laboratory where the weir can be calibrated in place. Where a formula must be applied to a weir, which cannot be directly calibrated, an accuracy of 2 per cent might be expected, if pains are taken in construction and use. There is an increasing doubt in the minds of engineers that the weir can be considered the precise measuring device that it was once thought to be.

According to Schoder, rounding the edge of the weir crest to a radius of about $\frac{1}{24}$ in. increases the flow about 2 per cent for a head of 0.5 ft., though the error diminishes to 0.5 per cent at a head of 1.35 ft. It is difficult to maintain a sharp crest, as rust, scale, algae growth, etc., may produce a rounding of about this amount. A radius of $\frac{1}{8}$ in. increases the flow 3 per cent; and $\frac{1}{4}$ in., $5\frac{1}{2}$ per cent, both at 0.5 ft. head. A change in the surface of the plate from a smooth polished sheet of metal to a roughness similar to that of a file also increases the discharge about 2 per cent. Changes in the velocity distribution in the channel may produce very large differences in the discharge for the same head on the weir.

82. Comparison of Weirs and Weir Formulas.—For accuracy, the best type of weir is the contracted rectangular weir with a very deep channel of approach so that the velocity of approach is negligible. All the weir formulas give practically the same values for such a case, and they differ more from each other as the velocity of approach becomes of increasing importance.

It is generally agreed that end contractions are a source of error, and hence this type of weir is to be avoided, if possible. However, certain physical settings sometimes make it necessary. It is probable that a contracted weir with negligible velocity of approach is better than a suppressed weir with an abnormally high velocity of approach.

The present information is that for weirs with sharp crests, polished faces, and small velocity of approach the Francis, Fteley and Stearns, or Rehbock formulas are suitable. For conditions that are not so ideal, the Bazin formula may be better.

In addition to sharp-crested weirs there are weirs with rounded crests or sloping faces or both and also with flat tops. The spillway of a dam is one variety of such a weir. For some types of

construction the coefficient is greater than for a standard weir, and for others it is less. No general rule can be established, but each type must be calibrated.

EXAMPLES

153. The width of the weir in Fig. 100 is 7.573 ft. Neglecting the velocity of approach, what is the rate of discharge when $H = 1.200$ ft.? If $Z = 2.85$ ft., find the rate of discharge by trial, using the Francis formula with $\alpha = 1$. *Ans.* 33.15, 34.0.

154. For the case in Prob. 153, find the value of y if $\alpha = 2$ and then solve by the modified Francis formula. *Ans.* 1.046, 34.8.

155. Solve Prob. 153 by the Fteley and Stearns formula. *Ans.* 34.15.

156. Solve Prob. 153 by the Bazin formula. *Ans.* 34.5.

157. Solve Prob. 153 by the Rehbock formula. *Ans.* 33.6.

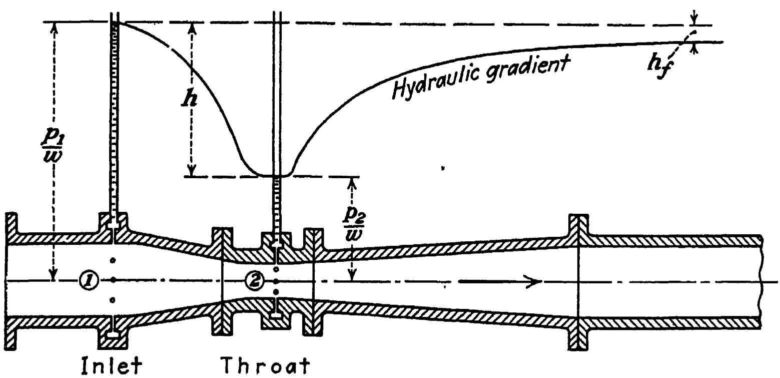


FIG. 111.—Venturi tube with converging entrance cone.

158. Assume that the weir in Fig. 102 is also 7.573 ft. in width. What would be the rate of discharge when $H = 1.200$ ft. by both the Francis and the Gourley and Crimp formulas? What would be the maximum value of H for which the Francis formula could be used? *Ans.* 32.1, 32.0.

159. What is the rate of discharge of a 54-deg. triangular weir when $H = 0.400$ ft.? With the same value of H what would be the rate of discharge of a 90-deg. triangular weir? (Use K from curves.) What would be the value of H for a rate of discharge of 2.0 cu. ft. per sec., if a 60-deg. triangular weir is used? (Use equation.) *Ans.* 0.1275, 0.2550, 1.15.

83. Venturi Tube.—The principle of this tube was investigated by Venturi, an Italian, about 1791. It was applied to the measurement of water by Clemens Herschel in 1886. As shown in Fig. 111, it consists of a tube with a constricted *throat* which produces an increased velocity, accompanied by a reduction in pressure, followed by a gradually diverging portion in which the

velocity is transformed back into pressure with but slight friction loss. Since there is a definite relation between the pressure

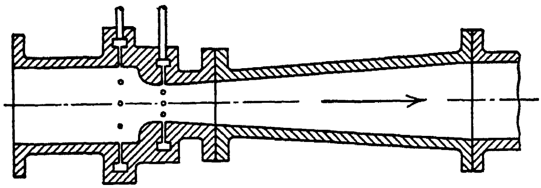


FIG. 112.—Venturi tube with entrance profile that of the standard nozzle of the Society of German Engineers.

differential and the rate of flow, the tube may be made to serve as a metering device.[1]

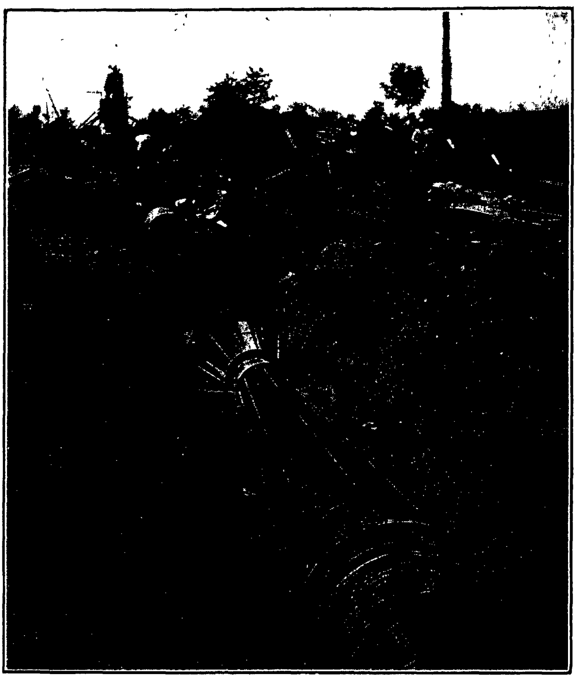


FIG. 113.—Venturi meter in wood pipe line. (*Courtesy of Builder's Iron Foundry.*)

The conventional form of venturi tube with a converging cone between inlet and throat is shown in Fig. 111, but in Fig. 112 is

[1] The venturi flume is identical in principle. It consists of a constricted section in an open channel. The flow is a function of the drop in the water surface at this section. The Parschal flume is a modification.

shown a new type that is preferred by some owing to the belief that the coefficient is less sensitive to changes in the smoothness of the surface. The profile for Fig. 112 might be that of the standard normal nozzle developed by the Society of German Engineers.

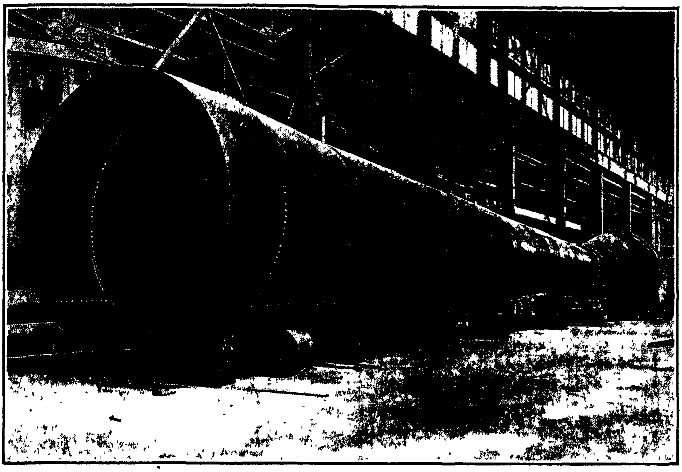


FIG. 114.—Venturi meter of riveted steel. (*Courtesy of Builder's Iron Foundry.*)

Equating H_1 to H_2, assuming that between (1) and (2) the friction loss is zero, we have, for the ideal case,

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g},$$

and, since $\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) = h$, whether the tube is horizontal, as shown in Fig. 111, or not, we have

$$\frac{V_2^2}{2g} - \frac{V_1^2}{2g} = h.$$

By the equation of continuity, $V_1 = (A_2/A_1)V_2$, and hence ideally[1]

[1] Strictly speaking, gravity does not have any effect in such a case as this where the flow is entirely confined. The value of g appears only because the differential pressure is expressed in terms of a height of fluid. It can be eliminated, since $gh = gp/w = p/\rho$. It is merely our system of units that brings g into the equation.

$$V_1 = \sqrt{\frac{2gh}{[(A_1/A_2)^2 - 1]}}, \qquad V_2 = \sqrt{\frac{2gh}{[1 - (A_2/A_1)^2]}}.$$

Since there is some friction loss between (1) and (2), the true velocities are slightly less than the values given by these expressions. Hence, when these velocities are multiplied by the corresponding areas, we shall introduce a coefficient c_d so that

$$q = \frac{c_d A_1}{\sqrt{(D_1/D_2)^4 - 1}}\sqrt{2gh} = \frac{c_d A_2}{\sqrt{1 - (D_2/D_1)^4}}\sqrt{2gh}. \qquad (94)$$

For a given tube, the dimensions are known and are constant; hence, the meter constant is

$$M = A_1/\sqrt{(D_1/D_2)^4 - 1} = A_2/\sqrt{1 - (D_2/D_1)^4}.$$

Thus Eq. (94) may be reduced to the simple form

$$q = c_d M\sqrt{2gh}. \qquad (95)$$

Although c_d actually varies with the flow, the variation is comparatively small, and, if a constant average value were assumed, the equation would reduce to

$$q = K\sqrt{h}. \qquad (96)$$

Unless specific information is available for a given tube, the value of c_d may be assumed to be about 0.99 for large tubes and about 0.97 or 0.98 for small ones provided the flow is such as to give reasonably high Reynolds numbers. These values will probably be accurate within about 1 per cent for new tubes or tubes newly cleaned. As the surface of the tube between inlet and throat becomes roughened in time, the increased friction will reduce the coefficient slightly. Also, a deposit of scale might change the areas a slight amount and affect the value of the coefficient. Venturi tubes in service for a number of years have shown a decrease in c_d of the order of 1 to 2 per cent.[1]

For a given venturi tube, c_d varies not only with the velocity but also with the viscosity and density of the fluid and hence is a function of Reynolds number. For the purpose of computing

[1] ALLEN, C. M., and L. J. HOOPER, "Venturi and Weir Measurements," *Mech. Eng.*, p. 369, June, 1935.
PARDOE, W. S., *Mech. Eng.*, p. 60, January, 1936.

this number, the diameter of the throat is used, since that dimension has more influence than the diameter of the inlet.

The value of expressing the coefficient as a function of Reynolds number is it makes it possible to select the proper value of c_d for

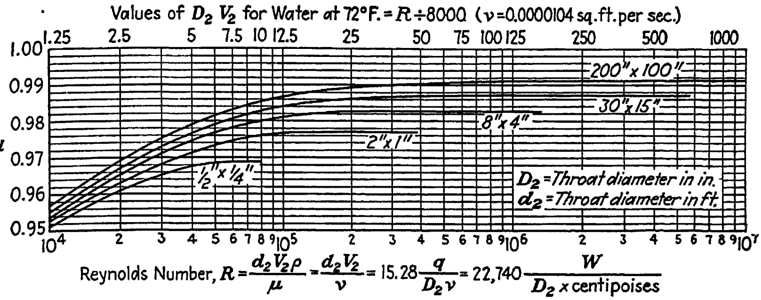


FIG. 115.—Coefficients for venturi tubes with a diameter ratio of 0.5.

any fluid whatever and does not restrict the use of the tube to the fluid for which it has been calibrated. Thus it is possible to calibrate a given tube with water, for example, and then use it for oil or air or anything else for which the kinematic viscosity is known.[1]

Values of venturi-tube coefficients as functions of both Reynolds number and size are shown in Fig. 115. This diagram is for a diameter ratio of 0.5 only, but values may be modified to suit other ratios by the aid of Fig. 116. For example, suppose it is desired to determine the coefficient for a 20 by 15 in. tube with a Reynolds number of 58,000. For that Reynolds number and throat diameter the 30 by 15 curve in Fig. 115 gives the value of c_d as 0.98. In Fig. 116 select the curve for which c_d is 0.98 at a diameter ratio of 0.5 and on this same curve at a diameter ratio of 0.75, the value of c_d will be found to be 0.974.[2] These dia-

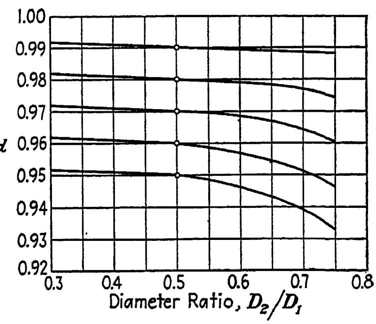


FIG. 116.—Variation of venturi tube coefficients with diameter ratio but with constant throat size and constant Reynolds number.

[1] For evidence on this matter see Fig. 122 and accompanying remarks.

[2] The data in Fig. 115 are a composite of the results obtained for venturi tubes of Builder's Iron Foundry and Simplex Valve and Meter Co. prepared

grams are for normal venturi tubes of the type shown in Fig. 111 with machined surfaces at the pressure measuring sections but, except for the very small sizes, with unmachined cast-iron cones. For very smooth cones the coefficients will be slightly higher and for other cone angles or for other forms, as in Fig. 112, there may be slightly different values, but in most cases it is believed that the values given will be accurate to within 1 per cent.

Occasionally, the calibration of a venturi tube has given a value for c_d of more than 1. Such an abnormal result is sometimes due to improper piezometer openings.[1] Again, it may be due to a spiral flow in the pipe line, which increases the pressure at the wall due to the centrifugal effect. By the principle of the free vortex (Art. 58) the rotation will be greater in the throat than at the inlet, and hence the excess of wall pressure over average pressure will be greater at the throat. Thus, although both pressure readings will be too high, the difference between them will be too low. Since the measured value of h is then less than its true value, the coefficient c_d will be correspondingly increased.

Reverting to Eq. (29) and applying it to the venturi tube with $\frac{p_1}{w} - \frac{p_2}{w} = h$, we have

$$\frac{(V_2^2 - V_1^2)}{2g} = h - [h_f + (N_2^2 - N_1^2) + (B_2^2 - B_1^2)].$$

For the ideal case of frictionless flow, each and every one of the terms in the brackets is zero. This would result in a value of 1 for the coefficient, but it is not necessary for the flow to be ideal to produce the same result. All that is necessary is for the sum of all the terms in the brackets to be zero. The friction h_f is always positive; but in the case of converging flow, which is what we have here, the values of the other two terms will generally be negative. Since the pressure across any cross section in a straight pipe is independent of the velocity distribution, it follows that $\frac{p_1}{w} - \frac{p_2}{w}$ has the same value for all streamlines. Therefore,

jointly by Ed S. Smith, Jr., and W. S. Pardoe for use in the fourth edition (1937) of the *A.S.M.E.* Fluid Meters Report. Figure 116 is a graphical representation of an empirical equation proposed by W. S. Pardoe.

[1] Allen, C. M., and L. J. Hooper, "Piezometer Investigation," *Trans. A.S.M.E.*, vol. 54, paper HYD-54-1, 1934.

if V_1' and V_2' indicate values on a streamline in idealized flow,
$$\frac{V_2'^2}{2g} - \frac{V_1'^2}{2g} = \frac{p_1}{w} - \frac{p_2}{w} = \text{constant}, \quad \text{or} \quad V_2'^2 = V_1'^2 + \text{constant}.$$
Graphically, V_2' is the hypotenuse of a right-angled triangle of which V_1' is one side while the other side is proportional to the square root of the pressure drop. Figure 117 shows the graphical construction for converting the velocity profile at (1) into that at (2) and shows that the percentage variation in velocity becomes much less. This figure is drawn to scale for a diameter ratio of 2:1. It is thus apparent that N_2^2 is less than N_1^2.

Also, as the stream contracts, its axial speed increases, but, since there is nothing to cause the transverse components, which constitute the turbulence, to increase, the flow straightens out

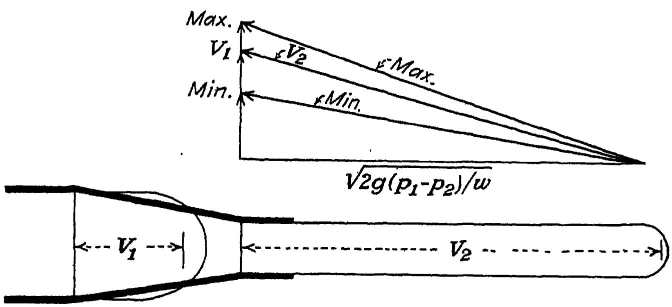


FIG. 117.—Effect of convergence upon velocity profile.

and becomes steadier. Thus for converging flow B_2^2 is less than B_1^2 in general. In cases where h_f is quite small, but where the initial values of N_1^2 and B_1^2 are large, it may be possible to have the entire bracket not only zero but even negative.

Although such a case is very unusual, it may be experienced at times. The reason for introducing this discussion here is that the venturi tube is one case where there may be considerable difference in the velocity profile and turbulence at the two sections, and it is desired to call attention to the fact that what is ordinarily termed friction loss in hydraulics may include other effects as well. It is in cases where the true friction is relatively small that these other factors become appreciable. Strictly speaking, true friction is found as a result of measuring the difference in H between two points only when the velocity distribution and the turbulence are the same at the two sections.

The venturi tube is an accurate device for the measurement of rate of discharge for all types of fluids and is most valuable for measuring large flows in pipe lines. By a suitable recording device it becomes a meter that will not only give a continuous record of the instantaneous rate of flow but will integrate it so as to give the total quantity. Outside the installation costs, its only disadvantage is that it introduces a permanent frictional resistance in the pipe line. This friction loss is practically all in the diverging part from (2) to (3) and is much larger than the friction between (1) and (2) (see Fig. 65). It varies with the construction of the tube but is ordinarily from 0.1 to $0.2h$, where h is the differential, in feet, of the fluid flowing.

The diameter of the throat was originally made one-third that of the inlet so that $A_2/A_1 = 1/9$. But in order to reduce the resistance offered by the tube and also in order to keep the throat pressure from becoming less than atmospheric in some cases, it is now quite common to use a throat ratio of $1/2$. In fact, values of D_2/D_1 range from $1/4$ to $3/4$. A small value of the ratio increases the value of the differential pressure for a given flow and increases the accuracy of the reading. But it is accompanied by a higher value of frictional resistance for the tube and may produce an undesirably low pressure at the throat, which would cause liberation of dissolved air or even vaporization of the liquid at that point. The only disadvantage of a large value of the ratio D_2/D_1 is that the smaller differential reading requires a more accurate means of measurement.

A venturi tube is rated by the size of the inlet. Thus, an 8-in. venturi is one that will go into an 8-in. pipe. If it is desired to express the throat diameter, also, the tube could be designated as an 8-by-4 or an 8-by-6 venturi, for example.

The angle of the cone at entrance to the throat is usually about 20 to 30 deg. The angle of the cone constituting the downstream portion is from 5 to 14 deg. It is believed that the minimum friction loss is obtained for an angle of about 5 deg. The only merit of a larger angle is that it decreases the length of the tube. The value of this angle has absolutely no effect on the coefficient for the tube. In fact, the discharge cone could be omitted altogether, and we should then have the nozzle type which is shown in Fig. 118. This is simpler than the venturi tube and can be installed between the flanges of a pipe line. It will answer the

same purpose, though at the expense of an increased friction loss in the pipe. Flow-nozzle coefficients are very similar to venturi-tube coefficients though not identical for some reason.

For accuracy in use, the venturi tube should be preceded by a straight pipe whose length is at least from 5 to 10 pipe diameters. If this is not possible, straightening vanes should be used to eliminate any rotation of the water in the pipe. The pressure should be read from piezometer rings, surrounding the pipe, with a number of suitable openings from them into the inlet and throat sections. A better arrangement might be to have a very narrow slot extend all the way around the circumference of the section. A pulsating flow tends to give a differential that is too high and therefore requires a value of c_d that is lower than the true value.

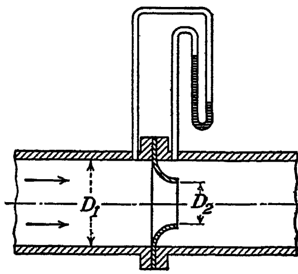


FIG. 118.—Flow nozzle.

EXAMPLES

160. Determine the value of K for a 12- by 4-in. venturi tube, assuming $c_d = 0.985$. *Ans.* 0.693.

161. If a differential manometer were used and h were replaced by y in. mercury (Fig. 16), find the equivalent value of K for the venturi tube in Prob. 160, if the fluid is (*a*) water at a temperature of 60°F., (*b*) oil with a specific weight of 50 lb. per cu. ft. *Ans.* (*a*) 0.709.

162. Suppose that the tube in Prob. 160 were to be used for a flow of 3.5 cu. ft. per sec. What would be the differential pressure in feet of the liquid, and what would be the probable frictional resistance offered by the tube?

163. Solve Probs. 160 and 162 for a tube with an inlet of 12 in. diameter and a throat of 6 in. diameter.

164. (*a*) A 12- by 6-in. venturi tube is to deliver 8 cu. ft. of water per sec. at a temperature of 60°F. What will be the value of K (see Table III)? (*b*) What would be the value of c_d for a venturi tube which is 8 by 6 in. but with the same flow as in (*a*)? *Ans.* 1.60, 0.981.

165. Suppose that the tube in Prob. 164 were to be used for an oil whose kinematic viscosity is 50×10^{-5} sq. ft. per sec. What would be the differential for a flow of 4 cu. ft. per sec.? *Ans.* 6.52 ft.

84. Orifice Meter.—An orifice in a pipe line, as in Fig. 119, may be used as a meter in the same manner as the venturi tube. It may also be placed on the end of the pipe so as to discharge a free jet. The coefficients are practically identical in the two

cases. The difference between the orifice in the present discussion and that in the earlier part of this chapter is that here the pipe walls are presumed to be nearer to the edge of the orifice, and thus complete contraction of the jet is interfered with. For the coefficients of Art. 69 to apply, the ratio D_0/D_1 should be about $\frac{1}{7}$ or less, where D_0 is the orifice diameter, and D_1 the pipe diameter. In practice the ratio is usually much higher than that, and thus the coefficient will tend to be larger than the values in Art. 69, and, since the degree of contraction is affected by the distance from the pipe walls, the contraction coefficient c_c will increase as the ratio D_0/D_1 increases. The velocity coefficient

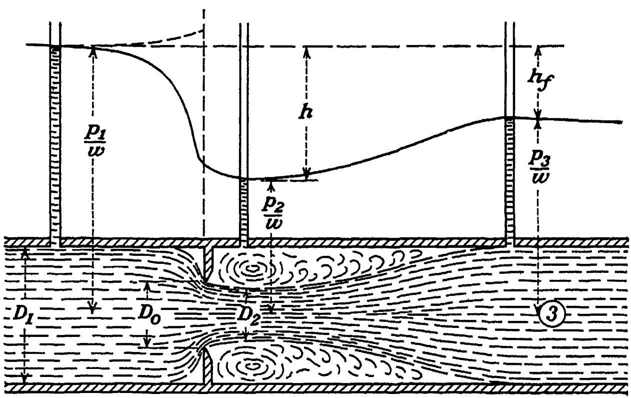


FIG. 119.—Thin-plate orifice.

is very close to unity and varies but little; hence, the orifice coefficient $c = c_c c_v$, as found in Eq. (61), will increase with increasing diameter ratio. This is shown in Fig. 120.

Instead of using Eq. (61), it is more convenient to employ the same equation as for the venturi tube, which is

$$q = \frac{c_d A_1}{\sqrt{(D_1/D_0)^4 - 1}}\sqrt{2gh} = \frac{c_d A_0}{\sqrt{1 - (D_0/D_1)^4}}\sqrt{2gh}, \quad (97)$$

where c_d is the ratio of actual to ideal rate of discharge as defined in Art. 65. The difference in the values for c and c_d may be seen in Fig. 120. Not only is Eq. (97) more convenient to use, but for diameter ratios up to 0.8 the value of c_d is much more nearly constant. Since it is not satisfactory to use higher ratios than 0.8, the variation beyond that point is of small importance.

It is also possible to reduce Eq. (97) to the simpler form

$$q = c_1 A_1 \sqrt{2gh} = c_2 A_0 \sqrt{2gh}, \tag{98}$$

where c_1 and c_2 are coefficients which include the velocity of approach factor, since each is the value of c_d divided by the corresponding square-root term. Their values are also shown in Fig. 120. If the value of any one of these coefficients is known for a given diameter ratio, corresponding values of the other three may readily be computed from it.

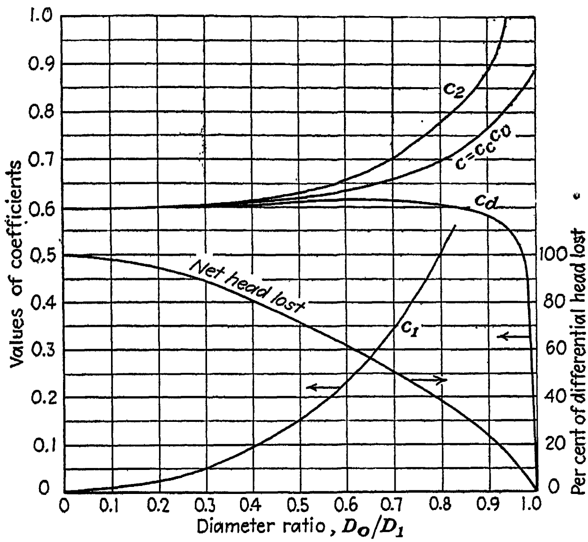


FIG. 120.—Sharp-edged orifices with pressure differential measured either at the flanges or at the vena contracta.

Values for c_d to a larger scale are shown in Fig. 121 and are reliable to about 1 per cent. Variations of that magnitude are to be found in comparing the data of different observers. This scatter is due to a number of factors, among which are variations in the finish of the orifice edge and the face of the orifice plate, differences in the roughness of the pipe, and also differences in the size of the pipe. For a small pipe the relative roughness is greater, even though the character of the surface is exactly the same. This tends to retard the velocity of flow along the pipe walls and along the face of the orifice plate. It is this velocity coming in from the side that produces the jet contraction. Any

reduction in this velocity will then reduce the jet contraction. Thus, if the coefficient for an orifice in a 15-in. pipe is 0.60, it may be 0.61 for the same diameter ratio in a 3-in. pipe.[1]

As in the case of the venturi tube, the equation may also be reduced to $q = c_d M\sqrt{2gh}$. The difference between a sharp-edged or square-edged orifice in a thin plate in a pipe line and the venturi tube or flow nozzle is that for both of the latter there is no contraction of the jet so that A_2 is also the area of the throat and is fixed, while for the orifice the area of the jet A_2 is, in general, less than the area of the orifice A_0 and is a variable. For the venturi tube or flow nozzle the coefficient is merely a velocity

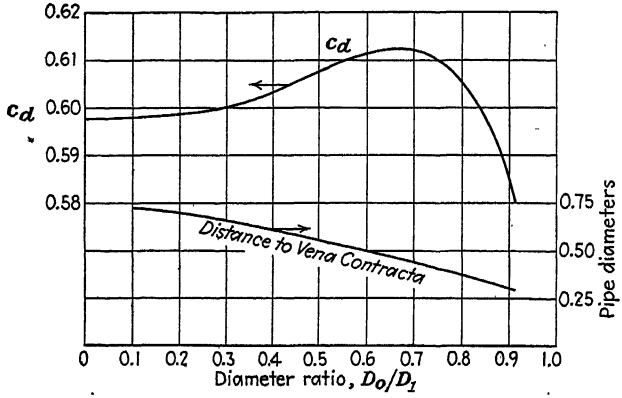


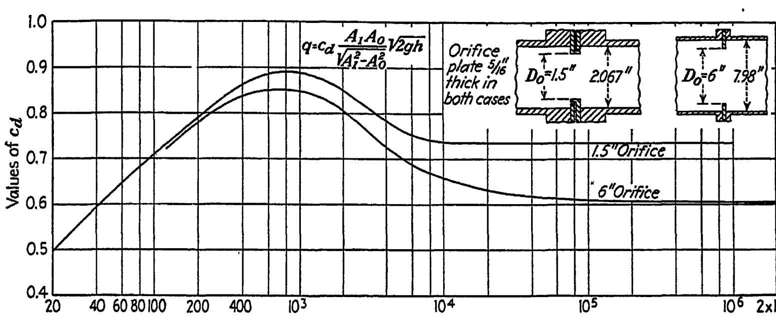
FIG. 121.—Coefficient for sharp-edged orifices.

coefficient, while for the orifice the value of c_d is much more affected by c_c than it is by c_v.

Not only is the coefficient c_d much less for the orifice than for the venturi tube, because of this jet contraction, but it varies in a different manner, as may be seen by comparing Figs. 115 and 122. The curves shown in Fig. 122 are from tests made by the author and are unique because of the unusually wide range of viscosities covered. Also, the data extended over a very large range of

[1] The curves shown in Figs. 120 and 121 were calculated by the author and based on data given by Beitler and Bucher, "The Flow of Fluids through Orifices in Six-inch Pipes"; Ed S. Smith, Jr., "Quantity-rate Fluid Meters," both in *Trans. A.S.M.E.*, vol. 52, no. 30, 1930; H. S. Bean, E. Buckingham, and P. S. Murphy, "Discharge Coefficients of Square-edged Orifices for Measuring the Flow of Air," *Res. Paper* 49, Bur. Standards; and *A.S.M.E.* Fluid Meters Rept., 1931.

values of Reynolds number. The fluids used were water and a series of oils up to a road oil, whose viscosity at the temperature in the test was 3,200 times that of water. For each fluid a number of different velocities were used, so that the curves represent points for all combinations of velocities and viscosities. All the test points were very close to the curves and definitely establish the fact that the coefficient varies with R in the manner shown. The curves for the two orifices do not coincide, because the two are not homologous. They were made from the same thickness of plate and were not beveled. Hence the thickness is relatively greater for the smaller orifice, as is shown in the sketch in Fig. 122. The effect of this is to reduce the jet contraction. This was veri-



Reynolds number, $R = \frac{d_o V_o \rho}{\mu} = 15.28 \frac{q}{D_o \nu} = 22{,}740 \frac{W}{D_o \times \text{centipoises}}$

FIG. 122.—Calibration of orifices with various fluids with viscosities of from 1 to 3,200 centipoises.[1]

fied later by tests in which different thicknesses of plate were used for the same-size orifice in the same pipe, and the coefficient was found to increase with increasing thickness of plate. To this must be added the fact that the coefficient also tends to be larger for a smaller pipe, for the reasons previously mentioned.

An explanation for the form of the curves shown in Fig. 122 may be had by considering R to be decreased by using a series of fluids of increasing viscosity but keeping all other factors the same. Starting with a low viscosity and a high value of Reynolds number, the coefficient is seen to be approximately constant. As

[1] DAUGHERTY, R. L., "Performance of Centrifugal Pumps when Pumping Oils," *Bull.* 130, Goulds Pumps Inc., Seneca Falls, N. Y., 1926. An experimental investigation made for Goulds Pumps and Union Oil Company jointly.

the viscosity increases, thus decreasing R, a point is reached at which the viscosity is sufficient to retard effectively the flow of the film of fluid over the upstream face of the orifice plate and thus to reduce the contraction of the jet. Further increases in viscosity continue to reduce the contraction until finally the size of the jet is equal to that of the orifice. That is, the contraction coefficient increases up to a value of unity, after which no further change is possible. But with increasing viscosity, and hence increasing fluid friction, the velocity coefficient continuously decreases from 1 for an ideal fluid with zero viscosity down to a value approaching zero as the viscosity of the fluid approaches infinity. The discharge coefficient, which is a function of these two, at first increases with decreasing R, since the influence of the contraction predominates, and reaches a maximum when contraction ceases. After that the change is due to the velocity coefficient alone, and so the discharge coefficient continuously decreases as R diminishes. For the venturi tube, on the other hand, there is no contraction at any stage; hence the coefficient for it is 1 for zero viscosity and decreases continuously with increasing viscosity or decreasing Reynolds number.

Although it may not seem so obvious, the same reasoning may be applied to changing any one of the other factors that enter into the value of R. Figure 122 also throws some light upon the variation of the orifice coefficients as given in Art. 69. It was there seen that the discharge coefficient decreased with increasing head up to a certain point, after which there was no appreciable change. Increasing the head on an orifice increases the velocity and hence increases R for the same fluid. Since none of the values in Art. 69 is for a low enough head or a high enough viscosity to give very low values of R, the values there given all apply to the right-hand portion of the curves in Fig. 122.

It was also stated in Art. 69 that the discharge coefficient decreased for increasing size of orifices. While the standard orifice has a sharp knife-edge or at least a square edge in a thin plate, these terms are purely relative. For a small orifice, slight imperfections in the edge and slight roughness of the plate become of more importance and hence tend to increase the coefficient above the value for the ideal sharp edge. Thus, although a large orifice may behave as if it had a mathematically sharp edge, a small one probably will not, even though the workmanship be

the same. The two orifices shown in Fig. 122 merely exaggerate the effect of size.

The values of the coefficients shown in Figs. 120 and 121 are all for large values of R where each coefficient becomes substantially constant for a given orifice and pipe.

In Fig. 119 is shown the pressure gradient, which reveals that the pressure drops to a minimum at the vena contracta at (2) and then gradually rises to (3), at a point from 4 to 8 pipe diameters downstream, after which it slowly diminishes owing to friction in the pipe. The drop in pressure between (1) and (2) is due to the acceleration given the stream in passing through the orifice, while the pressure drop between (1) and (3) is due solely to friction caused by the throttling effect of the orifice. The mechanism of this is that a considerable portion of the kinetic energy of the jet due to the velocity of translation is transformed into kinetic energy of rotation in vortices. This kinetic energy of rotation is not converted into pressure but is eventually dissipated by viscosity effects and converted into heat. The value of this is h_f.

The pressure differential may be measured between section (1), which is about one pipe diameter upstream, and the vena contracta, which is approximately one-half the pipe diameter downstream. The distance to the vena contracta is not a constant but decreases as D_0/D_1 increases, as shown in Fig. 121. The pressure differential may also be measured between the two corners on each side of the orifice plate and even between two points in the two faces of the plate itself, as in Fig. 122. These "flange connections" have the advantage that the orifice meter is self-contained; that is, the plate may be slipped into a pipe line between two pipe flanges and used without the necessity of additional work to make piezometer connections in the pipe. The pressure gradient in Fig. 119 shows the pressure variation in the center of the stream. In the corner where the orifice plate joins the pipe, the pressure is a little higher than in the center, owing to the centrifugal effect of the curved streamlines. For a similar reason, it should be expected that the pressure in the downstream corner would be a trifle less than that in the center of the pipe next to the orifice, but it is still higher than the pressure farther downstream at the vena contracta. Hence, the differential pressure measured with *flange connections* is not materially different from that measured between one pipe diam-

eter upstream and the vena contracta. Experimental data so far available do not indicate any appreciable difference between the coefficients obtained for the two sets of pressure connections. The difference, if any, seems to be much less than the scatter of the data due to other causes.

The pressure differential may be measured also between (1) and (3), but the value so obtained is much smaller. This is undesirable, except where it is necessary to employ some manometer with a limited capacity. This method has the merit that it is not very important just where the downstream piezometer is located, provided only that it is not too close. The same connections may then be used for different-sized orifices in the same pipe. However, since the roughness of the pipe has an appreciable effect, the entire setup should be calibrated. The coefficients are naturally larger than for the other connections. No values for them are given here.

The orifice as a metering device has the merit that it may be installed in a pipe line with a minimum of trouble and expense. Its principal disadvantage is the greater frictional resistance offered by it as compared with the venturi tube with the same diameter ratio. The friction added by the orifice may be found from Fig. 120.

EXAMPLES

166. Suppose that a 6-in. orifice were used in a 12-in. pipe to measure a flow of 3.5 cu. ft. per sec. What would be the differential? *Ans.* 12.6 ft.

167. What would be the head lost in Prob. 166, and what would it be for a venturi tube with the same differential? *Ans.* 9.24 ft., 1.26 to 2.52 ft.

85. Pitot Tube.—In 1730 Henri Pitot dipped a bent glass tube into the River Seine at Paris and found that the water rose up into the tube to a height that was proportional to the square of the velocity. Such a tube may thus serve to measure the velocity at a point in a stream and thereby enable the velocity distribution to be determined and the rate of discharge computed by Eq. (16).

The pitot tube is shown in (*b*) of Fig. 123. For an open stream or a jet, only the single tube is necessary; but for a closed pipe under pressure, it is necessary to measure the *static* pressure also, as shown by tube (*a*), and to subtract this from the total pitot reading to secure the differential pressure which gives the velocity head. The accurate determination of this static pres-

sure is the more difficult part of the problem, and it is the chief source of error in the use of the pitot tube.

The static pressure should preferably be measured at the wall of the pipe, as at (*a*), if the wall is smooth and the pipe straight, so that there will be no pressure variation across the section due to centrifugal action. The piezometer connection should also be upstream from the pitot tube, so as to avoid the disturbance of the flow which is caused by the insertion of the tube into the stream. A tube such as (*c*) projecting into the stream will give a reading that is much too low, owing to the acceleration experienced by the fluid in flowing around the end. The increased velocity at this point naturally causes a reduction in pressure at

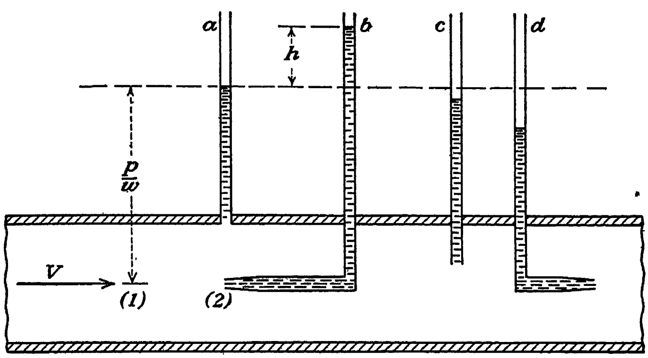


FIG. 123.

the end of the tube. A tube such as (*d*) directed downstream will give a still lower reading but not quite so much as $V^2/2g$ below the static pressure. Tubes (*b*) and (*d*) may be combined to form what is called the "pitometer" and will then give a differential that is larger than the velocity head h. Such an instrument must be calibrated to determine the value of its coefficient.

Obviously, the differential pressure may be measured by any suitable manometer arrangement and does not necessarily involve an actual column of the fluid whose velocity is being observed, such as is shown in Fig. 123 for illustrative purposes. In fact, if the fluid were air, for example, that would be impossible.

The derivation of the formula for the pitot tube is very simple, consisting merely in applying the energy equation to the *stream tube* leading up to the orifice. Thus, at a distance in front of the pitot, where the velocity is undisturbed, $H_1 = (p/w) + (V^2/2g)$,

while at the orifice, $H_2 = (p/w) + h$. Equating these two, $h = V^2/2g$, where h is identical with the pressure at the stagnation points in Figs. 52 and 56. It is the same as the dynamic pressure, or velocity pressure, defined in Art. 53 *c*.

For the case of a pitot tube pointed directly upstream and where there is no turbulence, the foregoing gives

$$V = \sqrt{2gh}. \tag{99}$$

This equation is not only a theoretical one; it is actually correct for the conditions specified. However, in most cases of flow there is more or less turbulence in the stream, which means that the velocities of individual particles fluctuate not only in magnitude but also in direction. Hence, in general, the velocities of the particles make some angle with the axial direction of the channel and with the axis of the pitot tube. We are usually less interested in the value of the instantaneous velocity V' than we are in the axial component V. Thus, if α is the average value of the angle that the velocities make with the axial direction, we desire the value $V = V' \cos \alpha$.

If the pitot tube is placed at an angle α in a stream flowing without any turbulence or, what is the same thing, is moved with a velocity V' through a still fluid but with its axis turned at an angle α, the pressure recorded by it will not be that due to V, the velocity component along the tube axis, but will be approximately $h = \cos \alpha \times V'^2/2g$. From this, $V' = \sqrt{2gh}/\sqrt{\cos \alpha}$. Since $V = V' \cos \alpha$,

$$V = \sqrt{\cos \alpha}\ \sqrt{2gh}. \tag{100}$$

The theory that is involved in this derivation is not exactly true, because the flow conditions assumed here are the same as those in finding the stagnation pressure in Fig. 52. This is correct as long as the axis of the tube is symmetrical with the flow, but, when the axis is turned through an angle, the streamlines are much altered from the symmetrical distribution shown. Also, in the latter case, the streamlines, and consequently the pressures recorded, depend somewhat upon the exact shape or form of the tip of the tube. In spite of these defects, Eq. (100) is a good approximation and agrees very closely with experimental values for all angles up to about 30 deg. It shows that the pitot-tube

coefficient varies with the turbulence and that it approaches unity as the turbulence diminishes. It also makes it clear that the coefficient is introduced in the pitot-tube formula not because Eq. (99) is not true but rather to correct for the flow conditions and to give the axial component of velocity.

There is some difference of opinion as to the magnitude of α for turbulent flow, but the general belief is that it is quite small and ranges from about 6 to 30 deg. according to the degree of turbulence. If the theory were precisely correct, and if the exact value of α were known in a given case, Eq. (100) could be applied. However, for the tubes normally used and for usual

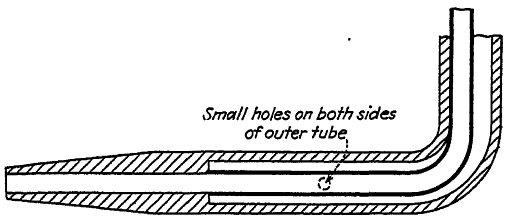


FIG. 124.—Pitot tube.

conditions encountered, present practice indicates that the true value of V is given very closely by[1]

$$V = 0.976\sqrt{2gh}. \tag{101}$$

However, if a pitot tube is used in a good jet, such as in Fig. 84, for example, it is proper to take the coefficient as unity. The preceding coefficient applies for the customary flow in pipes and similar cases.

Where conditions are such that it is impossible or impracticable to measure the static pressure at a wall, a combined tube such as that shown in Fig. 124 may be used. In this the static pressure is measured through two or more holes drilled through an outer tube into an annular space. The upstream end of the tube is pointed, and it is assumed that the flow follows along the outside of the tube so that the pressure is the same as that along the pipe

[1] MOODY, L. F., "Measurement of the Velocity of Flowing Water," *Proc. Eng. Soc. Western Penn.*, vol. 30, p. 319, 1914.

WHITE, W. M., and W. J. RHEINGANS, "Photoflow Method of Water Measurement," *Trans. A.S.M.E.*, vol. 57, no. 6, August, 1935.

COLE, E. S., "Pitot Tube Practice," in same issue. Discussion in *Trans. A.S.M.E.*, vol. 58, no. 2, February, 1936.

wall. Such tubes have been used very successfully by W. B. Gregory and others.

Inspection of Fig. 56 reveals that for a round-nosed body of revolution with its axis parallel to the flow, the velocity pressure $V^2/2g$ is found at the tip or stagnation point; the pressure then diminishes to zero, and becomes negative along the side. This same pressure variation is shown for the round-nosed Prandtl tube in Fig. 125. But the effect of the stem of the tube, at right angles to the stream, is to produce an excess pressure of $V^2/2g$ on its surface, and this pressure diminishes upstream, as shown. If the piezometer orifice in the side of the tube were located at the point where the excess pressure caused by the stem equaled the negative pressure caused by the flow around the tube, the

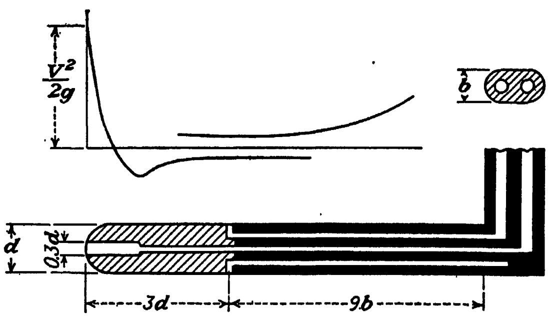


FIG. 125.—Prandtl-type pitot tube.

true static pressure would be obtained. Actually, there are a number of holes, usually eight, spaced around the entire circumference, and it is obvious that the pressure on the bottom of the tube cannot be influenced by the presence of the stem 180 deg. away. However, tubes constructed with the proportions indicated in Fig. 125 have been found to be quite reliable. A feature of this tube is that if it is not placed exactly in line with the flow, the error in the static reading and that in the pitot reading practically balance each other so that the differential will be measured accurately up to an angle of 15 deg. from the direction of the velocity.

A feature of the Prandtl type of tube is that it is relatively insensitive to direction. The type shown in Fig. 126 is just the reverse and is therefore called a direction-finding pitot tube. On examination of Fig. 56, it is seen that the pressure becomes zero

at some point along the surface. For two-dimensional flow past a small cylinder, whose axis is at right angles to the stream, this zero point is found experimentally to be at an angle of 39¼ deg. from the center line, giving a total included angle between (*a*) and (*b*) of 78.5 deg. If the tube is rotated about its axis until the pressure differential between (*a*) and (*b*) is zero, then it is known that the direction of the stream is just midway between those two points. If the pressure is then read for either (*a*) or (*b*) alone, its value will be the static pressure. If the tube is rotated 39¼ deg. from this position, the total pressure is obtained. Hence, such a tube determines direction quite accurately and also gives correct values for both static and total pressure. The velocity pressure is, of course, the difference.

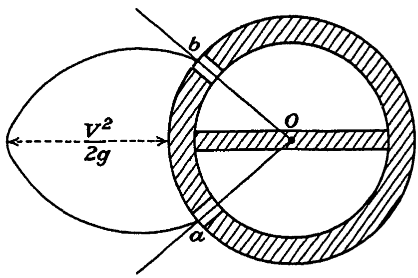


FIG. 126.—Direction-finding tube.

The pressure diagram shown in Fig. 126, like that in Figs. 56 and 125, is for the pressure relative to the static pressure. This type of tube has been used extensively in wind tunnels and has proved to be quite accurate in investigations made in the Hydraulic Machinery Laboratory of the California Institute of Technology for the pumping plants of the Colorado River aqueduct.[1]

86. Piezometer Connections.—The accuracy with which the rate of discharge may be determined by the devices previously described in this chapter depends not only upon the choice of a correct value for the coefficient of discharge but also upon the correct measurement of one or more pressures. Often the necessary value is a very small difference between two pressures, which makes accuracy all the more important. Gages and other instru-

[1] BINDER, R. C., and R. T. KNAPP, "Experimental Determinations of the Flow Characteristics in the Volutes of Centrifugal Pumps," *Trans. A.S.M.E.*, vol. 58, 1936.

ments have been described in the section on hydrostatics but, where the fluid is flowing, the piezometer connections also become important. Since these various metering instruments usually involve pressure measurements of fluids flowing with relatively high velocities, it seems appropriate to insert a brief discussion of the subject here.

The location for a piezometer connection should preferably be in a straight section of sufficient length for normal flow to have become established and one that is free of disturbing influences. The wall around the piezometer opening should be smooth and parallel to the stream flow. Unusual roughness or even waviness of the surface will produce deflections in the normal currents and may cause the pressure readings to be either too high or too low. In order to correct for such local deviations, it is desirable to have two or more connections around the periphery of the cross section. If the measurement of a pressure at a section where there is curved flow cannot be avoided, then it is essential to take readings for several points uniformly spaced around the periphery and to average them.

The piezometer opening in the side of the channel should be strictly flush with the surface. Any projection, such as that of (c) in Fig. 123, will result in a large negative error. Thus, Allen and Hooper[1] found that a projection of 0.1 in. will cause an error of 16 per cent, and 0.5 in. an error of 34 per cent, of the local velocity head. By local velocity is meant the velocity at the orifice. Even a slight burr will produce an error, and to insure against this it is advisable to round the entrance slightly.

Neither should there be a recess at the end of the piezometer tube. A slight recess produces a positive error, and a larger one a negative error, which vanishes as the length of the recess becomes two diameters or more. This indicates that whatever be the size of the orifice opening, it should be maintained at that size for at least two diameters.

The diameter of the orifice should be small relative to the channel dimensions, but, if it is too small, excessive throttling caused by it may be a source of error, since the observer may believe equilibrium has been attained when such is not the case. Also, any slight leakage from the tubing will then produce a marked effect on the readings. Allen and Hooper did not find

[1] Reference, p. 156.

any appreciable variation with orifices from $\frac{1}{16}$ to $\frac{27}{32}$ in. in diameter. However, the viscosity causes the fluid flowing past the orifice to exert a slight drag on the fluid within the tube and thus to lower its reading. This effect is said to diminish as the size of the hole decreases.[1] It is reported to reduce the reading by 1 per cent of the local velocity head, but evidently the effect of the size upon this is not great.

The axis of the tube must be normal to the surface for a distance of at least two diameters. If the tube is inclined so that its axis points slightly upstream, there will be a pitot-tube effect, even if the plane of the orifice is flush with the wall surface, so that the reading will be too high. Conversely, for the axis inclined in the opposite direction, the reading will be too low.

Where the pressure is pulsating, the fluctuations may be damped down by throttling for the sake of ease in reading. This will give a true mean value of the pressure, but in most of the fluid metering devices, the flow is a function of the square root of the pressure. For a pulsating flow the square root of the mean pressure is always greater than the mean of the square roots of the pressures, and it is the latter that should be used in the formulas. Hence, the value of the pressure, or the differential pressure, actually used is too large. Therefore, for such a flow the discharge coefficients should be further reduced below the values given. However, since the amount by which they should be reduced cannot well be predicted, it is necessary to eliminate the pulsations if accuracy is to be attained.

87. Other Methods of Measuring Discharge.—In addition to the devices used for the measurement of rate of discharge, which have already been described in this chapter, are numerous others of which a few will be briefly mentioned in this and the following article. An adequate description of the technique of their use would be beyond the scope of this book.

Chemical.—If a strong salt solution of known concentration c_1 is added at a small known rate q_1 to a stream, the flow in the latter may be determined merely by a chemical titration to find the increase in its salt concentration. If the initial value of the salt concentration in the main stream is c_0, and the value after the addition is c_2, the flow q can be found from the relation

[1] Prandtl-Tietjens, "Applied Hydro- and Aeromechanics," p. 227, McGraw-Hill Book Company, Inc., 1934.

$$q(c_2 - c_0) = q_1(c_1 - c_2). \tag{102}$$

This method has been found to be very accurate. It does not require any long length of uniform flow; in fact, the more the turbulence the better the mixing. It is therefore especially suitable in the very cases where other methods cannot readily be applied.[1]

Salt Velocity.—Electrodes are installed in the stream at two sections which are some distance apart, and at each station the electrical conductivity is measured as a function of time. A small quantity of a concentrated salt solution is injected all at once just above the first electrode, giving a record such as is shown in Fig. 127. The time required for the fluid to flow from one section to the next is obtained by dividing the areas, which represent the additional conductivity, into two equal parts, as shown. If the

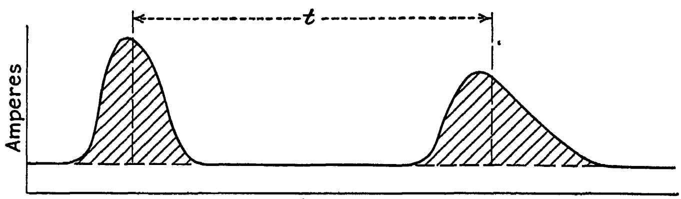


FIG. 127.—Salt-velocity method.

conduit is of uniform cross-sectional area, the mean velocity is the distance between stations divided by the time. If the conduit is not uniform, it is necessary to know the volume between the two stations. Then

$$q = \text{volume} \div \text{time}.$$

This method can be used either in open channels or in closed conduits. The greater the distance between sections the better, provided the distance is not so great that excessive diffusion will take place before the second section is reached. This method has been found to be very accurate.[2]

Thermal.—If the heat required to produce a known temperature rise in a flowing fluid is measured, and the specific heat is

[1] GROAT, B. F., "Chemihydrometry and Precise Turbine Testing," *Trans. A.S.C.E.*, vol. 80, p. 951, 1916.

[2] ALLEN, C. M., and E. A. TAYLOR, "The Salt Water Method of Water Measurement," *Trans. A.S.M.E.*, vol. 45, p. 285, 1923.

known, the weight of the fluid is at once determined. The heat is supplied by electricity so that the input may be accurately measured by a wattmeter.

88. Other Methods of Measuring Velocity.—The following are a few additional methods of measuring either the mean velocity or the velocity at a point. From either one the rate of discharge can be computed.

Gibson Method.—If a pressure-time diagram is taken while a valve is being closed, the appearance may be something like that in Fig. 128. Applying the principle in mechanics that

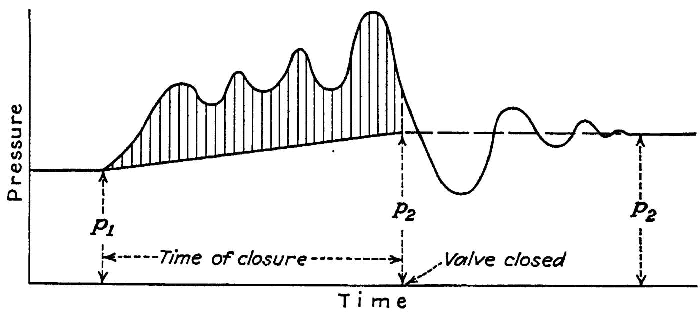


FIG. 128.—Gibson method.

impulse equals change in momentum or $F\ dt = m\ dV$ to this specific case,

$$\int F\,dt = \int pA\,dt \qquad \text{and} \qquad \int m\,dV = \left(wA\frac{L}{g}\right)(V - 0),$$

where V is the initial velocity which is reduced to zero with complete closure. Therefore,

$$\int p\,dt = \left(\frac{wL}{g}\right)V.$$

The value of $\int p\ dt$ is given by the area under the pressure-time curve and above the line joining p_1, the initial pressure, to p_2, the final static pressure. This line is not necessarily straight but is nearly so. Actual values of the equilibrium pressure for intermediate points may be found by noting that the area up to any point is proportional to the reduction of velocity up to that instant, while the difference between this pressure and p_2 is proportional to the square of the flow at that instant. This method

is naturally applicable only to pressure conduits but does not require the conduit to be uniform. The length must be at least 150 ft.[1]

Current Meter and Anemometer.—Unlike the two preceding methods, which determine mean velocity, these instruments measure velocity at a point only and hence require that a traverse of a cross section be made. When the instrument is designed for use in water it is called a *current meter,* and when it is for use in air it is called an *anemometer.* The two are the same in principle; but since the force exerted depends upon the density of the fluid as well as its velocity, the anemometer must be made so as to operate with very much less friction. The equation for either is practically

$$V = aN + b,$$

where N is revolutions per minute, and a and b are constants whose values can be determined only by calibration. However, the equation does not hold for very low velocities, and hence the instrument should not be used below a certain limiting speed, whose value depends upon the particular design.

Although there are many styles of current meters and anemometers, they all fall into two fundamental types. One is made up of a series of hemispherical or conical cups, which move in a circular path about an axis which is perpendicular to the flow; and the other is composed of vanes so that it resembles a propeller and rotates about an axis parallel to the flow. The former always rotates in the same direction regardless of the direction of the velocity so that it makes no distinction between positive and negative flow. In fact, its reading will be the same for any direction of flow that is in the plane of rotation, and it will even rotate, though at a reduced rate, for a flow that is at right angles to the plane of rotation. Thus this type of meter is not suitable for use where there are eddies and other irregularities in the flow. The latter type will give a reading that is more nearly proportional to the component of velocity along its axis and will rotate in the opposite direction for a negative flow. However, it usually has more friction in its bearings and is thus not so sensitive for low velocities.

[1] GIBSON, N. R., "The Gibson Method and Apparatus for Measuring the Flow of Water in Closed Conduits," *Trans. A.S.M.E.*, vol. 45, 1923.

The current meter is extensively used in natural streams and also in artificial open channels.

Hot-wire Anemometer.—This device is useful not only for indicating velocity at a point but for measuring the fluctuation in velocity, since it responds very rapidly to any change. It depends upon the fact that the electrical resistance of a wire depends upon its temperature, that the temperature, in turn, depends upon the heat transfer to the surrounding fluid, and that the coefficient of heat transfer increases with increasing velocity.

89. Compressible Fluids.—Strictly speaking, most of the equations that have been presented in the preceding part of this chapter apply theoretically only to incompressible fluids; but actually they may be used for all liquids and even for gases and vapors in cases where the pressure differential is small relative to the total pressure. Since this is the condition usually encountered in the metering of fluids, the preceding treatment has extensive application. But in order to give a complete picture of the subject, the modifications imposed by the consideration of compressibility will be briefly presented.

As in the previous cases, we shall derive equations for an ideal flow and then introduce a coefficient in order to obtain a correct result. The ideal conditions that will be imposed for a compressible fluid are that it shall be frictionless and that there shall be no transfer of heat or, in other words, that the flow shall be adiabatic. Because of the variation in density with both temperature and pressure, it seems better to express the rate of discharge in terms of weight rather than volume. Also, it is necessary to use the continuity equation in the form

$$w_1A_1V_1 = w_2A_2V_2.$$

The z terms will be omitted, since the change in elevation is usually zero or at least negligible in these metering devices.

From Eq. (32) the velocity in the throat of a venturi tube or of the jet from an orifice is

$$V_2 = c_v\sqrt{\frac{2g \times 778(E_1 - E_2)}{1 - \left(\frac{w_2A_2}{w_1A_1}\right)^2}}, \tag{103}$$

and the rate of discharge is

$$W = c_d w_1 A_1 \sqrt{\frac{2g \times 778(E_1 - E_2)}{\left(\frac{w_1 A_1}{w_2 A_2}\right)^2 - 1}} = c_d w_2 A_2 \sqrt{\frac{2g \times 778(E_1 - E_2)}{1 - \left(\frac{w_2 A_2}{w_1 A_1}\right)^2}}, \tag{104}$$

where A_2 is the area of the throat of the venturi tube or the area of the orifice, not that of the vena contracta. In these equations E_2 is the value after a frictionless adiabatic expansion from p_1 to p_2 and may be obtained by the aid of suitable vapor tables or charts, if the fluid is a vapor; or may be computed from the temperature or otherwise, if the fluid is a gas. Some knowledge of thermodynamics is required in order to obtain these values, but the presentation of that subject is outside the scope of this book.

However, the results can be expressed directly in terms of the pressures involved by employing Eq. (34). Since $h_f = 0$ for the ideal case,

$$\frac{V_2^2 - V_1^2}{2g} = \int_{p_2}^{p_1} v\,dp,$$

which may be integrated, since pv^k = constant, where k is the ratio of the specific heat at constant pressure to that at constant volume and ranges from 1.2 to 1.4 for most gases and vapors. If k varies, as it does for real gases and especially vapors, an average value must be used. The result is

$$\frac{V_2^2 - V_1^2}{2g} = \frac{k}{k-1} p_1 v_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{(k-1)}{k}}\right]. \tag{105}$$

By the aid of the continuity equation either one of the velocities may be determined. Thus,

$$V_2 = c_v \sqrt{2g \frac{k}{k-1} \frac{p_1 v_1 \left[1 - (p_2/p_1)^{\frac{(k-1)}{k}}\right]}{1 - (A_2/A_1)^2 (p_2/p_1)^{\frac{2}{k}}}}. \tag{106}$$

The rate of discharge may be shown to be

$$W = c_d A_1 \sqrt{w_1 2g \frac{k}{k-1} \frac{p_1 \left[1 - (p_2/p_1)^{\frac{(k-1)}{k}}\right]}{(p_1/p_2)^{\frac{2}{k}} (A_1/A_2)^2 - 1}}. \tag{107}$$

$$W = c_d A_2 \sqrt{w_2 2g \frac{k}{k-1} \frac{p_2\left[(p_1/p_2)^{\frac{(k-1)}{k}} - 1\right]}{1 - (p_2/p_1)^{\frac{2}{k}} (A_2/A_1)^2}}. \qquad (108)$$

These equations are too complex for convenient use in metering, and also it is easier to measure a pressure difference than a pressure ratio. Hence, they are replaced by the simpler equations for an incompressible fluid but with the insertion of a correction factor Y which compensates for the expansion.

For an incompressible fluid, the differential $h = (p_1 - p_2)/w$; and from Art. 83,

$$W' = wq' = c_d A_1 \sqrt{w \times 2g \frac{p_1 - p_2}{(D_1/D_2)^4 - 1}}$$

$$= c_d A_2 \sqrt{w \times 2g \frac{p_1 - p_2}{1 - (D_2/D_1)^4}} = c_d M \sqrt{w \times 2g(p_1 - p_2)}. \qquad (109)$$

Where the fluid expands as the pressure drops from p_1 to p_2, the specific weight will decrease from w_1 to w_2; but whichever one of these two values is used in Eq. (109), the quantity W' will be found to be larger than the true value W. The factor Y is defined as W/W' and may have two values depending upon whether w_1 or w_2 is employed. Therefore,

$$W = c_d M Y_1 \sqrt{w_1 2g(p_1 - p_2)} = c_d M Y_2 \sqrt{w_2 2g(p_1 - p_2)}. \qquad (110)$$

Values of Y may be found by experiment or by computing W by Eq. (107) or (108) and finding the relation between it and W' for a variety of conditions. The values so determined may then be used for all future work. It has been shown by Ed S. Smith, Jr., that values of Y may be plotted as very simple functions of a dimensionless ratio $\left(\frac{p_1 - p_2}{kp_1}\right)$. Such a chart for perfect gases is shown in Fig. 129. Slightly different values would be found for steam, for example. For a venturi tube the coefficient c_d is the same for a compressible as for an incompressible fluid, but for the orifice the coefficient c_d increases as the amount of expansion increases, owing to the fact that the jet no longer contracts in the same degree as for a liquid. Hence, in Fig. 129 the values of Y_1 for the orifice are somewhat larger than for the venturi tube.

Equation (105) also applies to such a device as the pitot tube. Let (1) refer to the values where the flow is undisturbed, while

(2) refers to the values at the pitot-tube orifice. The velocity pressure, or stagnation pressure, is then $p_2 - p_1$; and the velocity of the undisturbed fluid is

$$V_1 = \sqrt{2g\frac{k}{k-1}\frac{p_1}{w_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{(k-1)}{k}} - 1\right]}. \qquad (111)$$

Of course, a proper coefficient for turbulent flow may be introduced in just the same way as in Art. 85.

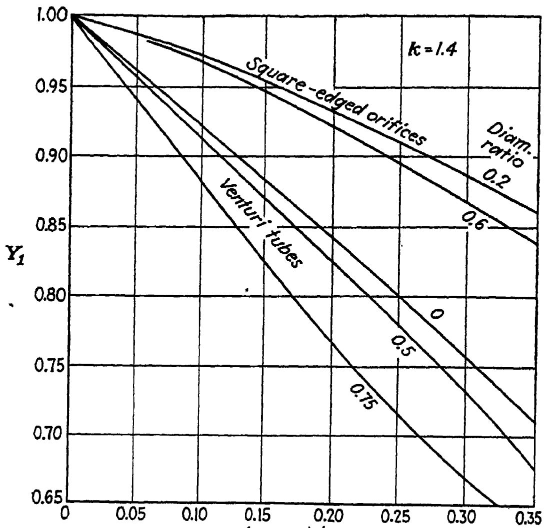


FIG. 129.

Equation (111) may also be used to give the velocity pressure for any particular velocity V_1. Rearranging the equation, the result is

$$p_2 - p_1 = p_1\left[\left(\frac{w_1(k-1)}{p_1 k}\frac{V_1^2}{2g} + 1\right)^{\frac{k}{(k-1)}} - 1\right]. \qquad (112)$$

Expanding this by the binomial theorem,

$$p_2 - p_1 = w_1\frac{V_1^2}{2g}\left[1 + \frac{w_1}{2p_1 k}\frac{V_1^2}{2g} + \cdots\right]. \qquad (113)$$

Since the velocity of sound in an ideal gas is

$$C = \sqrt{gkpv} = \sqrt{\frac{gkp}{w}},$$

the foregoing may be written as

$$p_2 - p_1 = w_1\frac{V_1^2}{2g}\left[1 + \frac{V_1^2}{4C^2} + \cdot \cdot \cdot \right]. \tag{114}$$

From this the amount of error involved in neglecting the compressibility may readily be determined. Thus, for air at 70°F., $C = 1{,}130$ ft. per sec.; and the error is 1 per cent when $V_1 = 226$ ft. per sec. This justifies the neglect of compressibility in ordinary problems in aeronautics, but it is also seen that for much higher velocities the effect is not negligible.

EXAMPLES

168. Assume that air at 70°F. flows through a venturi tube with a pressure at inlet of 100 lb. per sq. in. abs. and at the throat of 60 lb. per sq. in. abs. The inlet area is 0.60 sq. ft., and the throat area is 0.15 sq. ft. Assume the tube coefficient to be 0.98, and that for air $k = 1.40$. At this temperature and pressure the specific weight of air is $w_1 = 0.509$ lb. per cu. ft. (*a*) What is the value of Y_1? (*b*) What is the rate of discharge?

Ans. 0.743, 49 lb. per sec.

169. What would be the values of Y_1 and the rate of discharge for a square-edged orifice for the same data as in Prob. 168?

Ans. 0.88, 58 lb. per sec.

170. What is the value of the throat velocity in Prob. 168?

Ans. 922 ft. per sec.

171. A pitot tube is used for measuring air velocity where the initial pressure is 20 lb. per sq. in. abs., and the total tube pressure is 25 lb. per sq. in abs. That is, the velocity pressure is 5 lb. per sq. in. The air temperature is 70°, and therefore the specific weight is $w_1 = 0.102$ lb. per cu. ft. Find (*a*) velocity by the *hydraulic* formula, (*b*) true velocity.

Ans. 672 ft. per sec., 646 ft. per sec.

172. Suppose that air at the initial temperature and pressure in Prob. 171 were flowing with a velocity that was one-half that of a sound wave in the same medium; what would be the velocity pressure?

Ans. 3.73 lb. per sq. in.

90. Supersonic Velocity.—Whenever the velocity of a gas or a vapor equals or exceeds that of sound in the same medium, a decided change takes place in the flow pattern, and certain of the previous equations may no longer hold. Thus, in the case of flow through an orifice, the maximum possible jet velocity is that of a sound wave which is accompanied by a certain minimum

pressure which cannot be further diminished no matter how low the pressure in the space into which the jet discharges. The explanation of this is that a pressure wave, which is the same thing as a sound wave, cannot travel upstream any faster than the velocity with which the particles of the fluid travel downstream and that therefore the lower pressure cannot be transmitted up to the plane of the orifice. Hence, equations such as (103) to (110), inclusive, are to be used only so long as p_2 is greater than p_c and V_2 is consequently less than C. For all values of p_2 less than p_c, the jet velocity remains constant at the acoustic value, which is, for a perfect gas,

$$C = \sqrt{gkp_cv_c} = \sqrt{gkRT_c}, \tag{115}$$

where p_c and v_c are the *critical* pressure and specific volume, respectively; R is the gas constant; and T_c is the absolute temperature.

Substituting p_c for p_2 in Eq. (106) and equating V_2 to C in Eq. (115), we obtain

$$p_c^{\frac{(k-1)}{k}} = \frac{2}{k+1}p_1^{\frac{(k-1)}{k}} + \frac{k-1}{k+1}\left(\frac{A_2}{A_1}\right)^2 \frac{p_c^{\frac{(k+1)}{k}}}{p_1^{\frac{2}{k}}}, \tag{116}$$

which can be solved by trial. Fortunately, the last term, which covers the velocity of approach, is relatively small. If it is omitted, as it may be where the velocity of approach is negligible, this reduces to the simple relationship

$$p_c = p_1\left(\frac{2}{k+1}\right)^{\frac{k}{(k-1)}}. \tag{117}$$

For a gas, k is the ratio of the specific heat at constant pressure to that at constant volume and for air and diatomic gases ranges from 1.40 to 1.23 as the temperature increases. For superheated steam, k may be taken as about 1.3; and for wet steam, it is about 1.13. As long as the velocity of approach is negligible, it may be seen that p_c ranges from $0.528p_1$ to $0.58p_1$.

Since for the acoustic velocity and also for supersonic velocities, p_c may replace p_2 in the equations of the preceding article, and since p_c may be expressed in terms of p_1, it follows that the jet velocity and the rate of discharge are functions of p_1 only. Thus, if the velocity of approach is negligible, Eq. (106) becomes

$$V_2 = c_v\sqrt{2g\left(\frac{k}{(k+1)}\right)p_1v_1}, \tag{118}$$

and the rate of discharge is

$$W = c_dA_2\sqrt{2gw_1p_1\left(\frac{k}{(k+1)}\right)\left(\frac{2}{(k+1)}\right)^{\frac{2}{(k-1)}}}. \tag{119}$$

If the velocity of approach is not negligible, then it is necessary to compute C, which is the same as V_2, by Eq. (115) after determining p_c by Eq. (116). Equation (119) assumes that the gas follows the perfect-gas law $pv = RT$. If it does not do so, then the actual specific weight at pressure p_c should be used in conjunction with V_2 and A_2.

In order that the velocity of a stream may exceed that of sound, it is necessary to turn from the orifice to a nozzle which converges and then diverges. If the terminal pressure is below p_c, the velocity in the throat will be that of a sound wave in the medium, whose pressure is p_c and whose specific weight is w_c. The area of the throat thus determines the capacity of the nozzle. In the diverging portion the gas or vapor continues to expand to lower pressures, and the velocity continues to increase. If the terminal area is sufficiently large, the fluid can expand completely to the terminal pressure, no matter how small it may become. The velocity of the jet that is so obtained will then be much greater than the acoustic velocity at the throat. For the jet from a proper nozzle, the equations in the preceding article apply when the terminal pressure has any value whatever.

The venturi tube is seen to be a converging-diverging device and thus may enable supersonic velocities to be attained. In Fig. 130, if the pressure at the throat (2) is greater than p_c, a gas or a vapor will flow through the tube in much the same manner as a liquid. That is, the velocity will increase to a *maximum* value at the throat, which will be *less*, however, than the acoustic velocity; and then from (2) to (3) the velocity will *diminish*, while the pressure *increases*, as shown by the hydraulic gradient ABD. The pressure at (3), as represented by the height to D, is only slightly less than p_1 at A. If p_1 remains constant while the pressure at (3) decreases a little, the hydraulic gradient ABD will change to a similar one but with slightly lower values. As the pressure at (3) continues to decrease, p_1 remaining constant,

the pressure at (2) continues to diminish, and the velocity to increase, until the limiting acoustic velocity is reached, when the pressure gradient is *ACE*. If the pressure is further reduced to *H*, the pressure gradient is *ACFGH*, the jump from *F* to *G* being a pressure shock, similar to the hydraulic jump or standing wave often seen in open channels conveying water. As the pressure at (3) continues to diminish, the terminal pressure finally becomes H''' so that the pressure history is $ACFH'''$. In contrast with conditions for flow below the acoustic velocity, the velocity in the last case continuously increases from (1) to its *maximum* value at (3), which is much *greater* than the acoustic

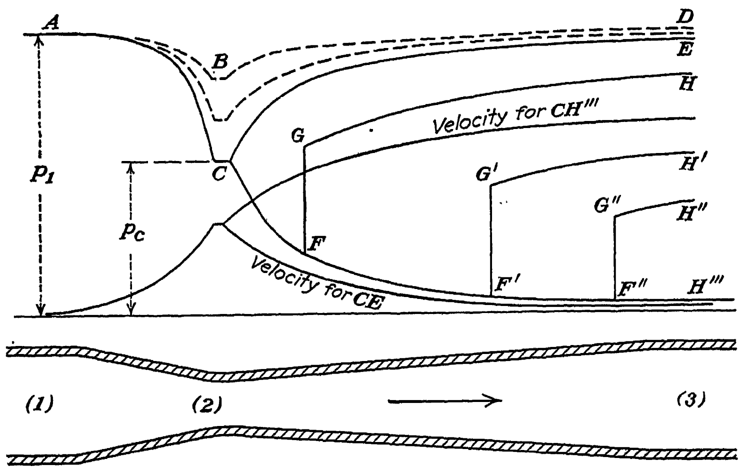


FIG. 130.—Compressible fluid in venturi tube.

velocity, while the pressure *continuously* decreases from (1) to (3).

As long as p_1 remains unchanged, the value of p_c, and hence of the throat velocity, remains constant, and thus the rate of discharge through the venturi tube is unaffected by any further decrease in the pressure at (3). The value of the pressure at (3) merely determines the velocity that may be attained at that point and the necessary area of the terminal cross section.

If p_1 is increased, the acoustic velocity may be shown to remain unaltered, but, since the density of the gas is increased, the rate of discharge is greater. The orifice and the venturi tube are alike in so far as discharge capacity is concerned. The only difference is that with the venturi tube, or a converging-diverging nozzle, a supersonic velocity may be attained at discharge from

the device, while with the orifice the acoustic velocity is the maximum value possible at any point.

It may also be added at this point that, when the velocity of a body through any fluid, whether a liquid or a gas, exceeds that of a sound wave in the same fluid, the flow conditions are entirely different from those for all velocities lower than this value. Thus, instead of streamlines such as are shown in Figs. 52 and 56, the conditions would be as represented in Fig. 131. A series of compression waves or shock waves travel along with the body and make an angle with the direction of motion which is a function of the ratio C/U, where C is the acoustic velocity, and U the

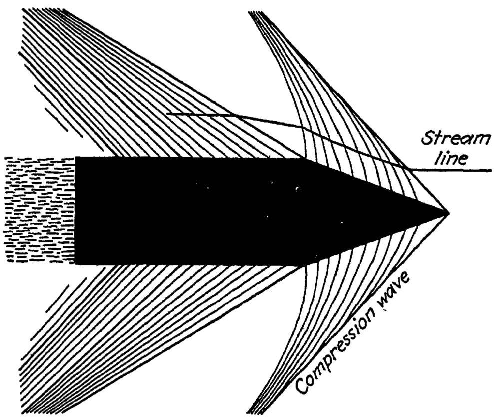


FIG. 131.—From photograph of projectile with velocity 1.576 times acoustic velocity.

velocity of the body or of the fluid at an infinite distance relative to the body. A streamline is unaffected by the body until it intersects a wave front, when it is abruptly changed in direction. This is because the body travels faster than the disturbance can be transmitted ahead.

EXAMPLES

173. For a perfect gas, if $k = 1.4 =$ constant, $p_1 = 14{,}400$ lb. per sq. ft., $R = 53.34$, $t_c = 70°$F., find the critical pressure and the acoustic velocity, neglecting the velocity of approach.

Ans. 52.8 lb. per sq. in., 1,130 ft. per sec.

174. What will be the rate of discharge for the case in Prob. 173 if the orifice diameter is 2 in., discharge coefficient is 0.60, and the jet discharges into a space where the pressure is 30 lb. per sq. in. abs.? What would be the result if the jet discharged into a perfect vacuum?

Ans. 3.98 lb. per sec.

175. Find the critical pressure and the acoustic velocity for wet steam flowing through an orifice with an initial pressure of 100 lb. per sq. in. abs., neglecting velocity of approach and any friction losses. (If the steam is initially dry and saturated, the use of steam tables or charts will give v_c = 7.15 cu. ft. per lb.) *Ans.* 58 lb. per sq. in., 1,475 ft. per sec.

176. Suppose that the data in Prob. 173 are applied to a venturi tube (or a nozzle) where the throat diameter is 2 in. and the terminal pressure is 5 lb. per sq. in. abs. Neglecting friction losses and velocity of approach, what is the rate of discharge? What is the maximum velocity? What should be the terminal area? (v_3 = 20.05.)

Ans. 6.63 lb. per sec., 2,100 ft. per sec., 0.0632 sq. ft.

177. Dry saturated steam at 100 lb. per sq. in. abs. flows through a suitable nozzle which expands it to 5 lb. per sq. in. abs. If k = 1.13 and steam tables or charts give v_1 = 4.43, find the velocity attained.

Ans. 3,230 ft. per sec.

178. For the case in Prob. 177, steam tables or charts give the values E_1 = 1186.5 B.t.u. per lb. and E_2 = 981.5 B.t.u. per lb. Find the jet velocity using these quantities. (Note that Eq. (103) is of general application, since the term in the denominator is merely a correction for the velocity of approach. In this problem the velocity of approach may be considered as negligible or A_1 is infinite.)

CHAPTER VIII

FRICTION LOSSES IN PIPES

91. Laminar and Turbulent Flow.—That there are two distinctly different types of fluid flow was demonstrated by Osborne Reynolds in 1883. He injected a fine threadlike stream of colored liquid at the entrance to a large glass tube through which water was flowing. When the velocity of flow in the tube was small, this colored liquid was visible as a straight line throughout the length of the tube, thus showing that the particles of water moved in parallel straight lines. But, as the velocity of the

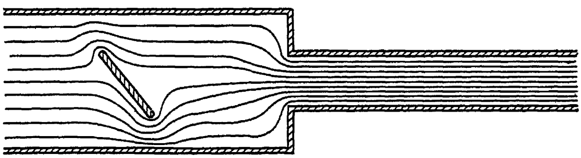


FIG. 132.—Laminar or streamline flow.

water was gradually increased by permitting a greater quantity to flow through the tube, there was a point at which the flow abruptly changed. It was then seen that, instead of a single straight line, the particles of the colored liquid were flowing in a very irregular fashion and forming numerous vortices. In a short time the color was diffused uniformly throughout the tube so that no streamlines could be distinguished. Later observations have shown that in this type of flow the velocities and pressures are continuously subject to irregular fluctuations.

The first type is known as *laminar*, *streamline*, or *viscous* flow. The significance of these terms is that the fluid appears to move by the sliding of layers or laminations of infinitesimal thickness relative to adjacent layers; that the particles move in definite and observable paths or streamlines, as in Fig. 132; and it is also a flow that is characteristic of a viscous fluid or at least a flow in which viscosity plays a significant part (see Fig. 1 and Art. 5).

The second type is known as *turbulent* flow and is illustrated in Fig. 133, where (*a*) represents the irregular motion of a large

number of particles during a very brief time interval, while (*b*) shows the erratic path followed by a single particle during a longer time interval. A distinguishing characteristic of turbulence is its irregularity, there being no definite frequency, as in wave action or any observable pattern, as in the case of eddies.

Large eddies and swirls and irregular movements of large bodies of fluid, which can be traced to obvious sources of disturbance, do not constitute turbulence but may be described as a disturbed flow. By contrast, turbulent flow may be found in what appears to be a very smoothly flowing stream and one in which there is no detectable source of disturbance. The fluctuations of velocity and pressure are furthermore comparatively small and can often be detected only by special means of observation.

At a certain instant, when the particle in Fig. 133 (*b*) is at the point *O*, it may be moving with the velocity *OD*, but the axial component *OB* is the value with which we are usually more

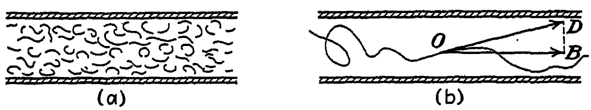


FIG. 133.—Turbulent flow.

concerned. Since in turbulent flow *OD* varies continuously both in direction and in magnitude, it is obvious that *OB* must likewise fluctuate in magnitude. However, in mean steady flow, there will be a constant mean value of the velocity at any fixed point.

92. Critical Reynolds Number.—If the drop in pressure in a given length of horizontal pipe is measured at different values of the velocity, it will be found that, as long as the velocity is low enough to secure laminar flow, the pressure difference will be directly proportional to the velocity, as shown in Fig. 134. But with increasing velocity, at some point *B*, where visual observation would show that the flow changes from laminar to turbulent, there will be an abrupt increase in the rate at which the pressure drop varies. If the logarithms of these two variables are plotted on linear scales, or if the values are plotted directly on logarithmic cross-sectional paper, it will be found that, after a certain transition region has been passed, a line will be obtained with a slope ranging from about 1.72 to 2.00. The lower value is found for pipe with very smooth walls, and with increasing roughness the slope increases up to the maximum of 2.00.

If the velocity is gradually reduced from a high value, the line CB will not be retraced. Instead, the points will lie along the curve CA. The value at B is known as the *higher* critical value, and that at A as the *lower* critical value. However, as Reynolds demonstrated, the velocity alone is not the deciding factor. Instead, the true criterion is the value of $dV\rho/\mu$, which is Reynolds number, as derived in Art. 59.

The upper critical value of Reynolds number is really indeterminate and depends upon the care taken to prevent any initial

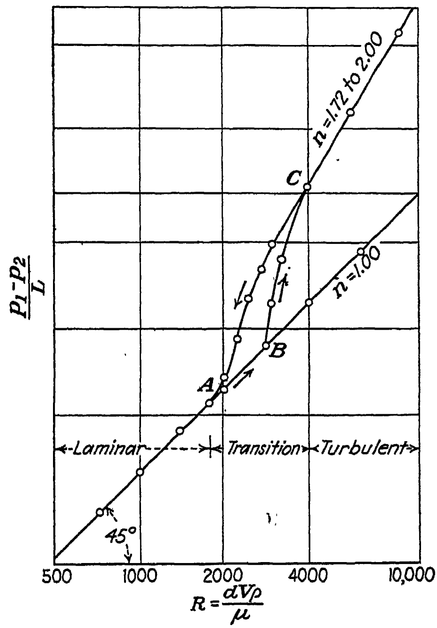


FIG. 134.—Critical Reynolds number.

disturbance from affecting the flow. Laminar flow has been maintained up to values of R as high as 50,000, but this type of flow under such conditions is inherently unstable. Normally, laminar flow is not to be expected at values of R above about 3,000. On the other hand, it is practically impossible for turbulent flow to persist at values of R below about 2,000, because any disturbance that is set up will be damped out. This lower value is much more definite than the other and is really the dividing point between the two types of flow. Hence it may be defined as the true critical Reynolds number.

Although the lower critical value is much more definite than the higher value, it is still subject to slight variations. Its value will be higher in a converging pipe and lower in a diverging pipe than in one of uniform diameter. Also, its value will be less for flow in a curved pipe than in a straight one. Even for a straight pipe the value may be as low as 1,000 where there is an unusual degree of roughness of the pipe wall. However, for the normal case of flow in a straight pipe of uniform diameter and normal surface roughness, the critical value may be taken as $R_c = 2{,}000$.

For water at 75°F., at which temperature the kinematic viscosity is 1.00×10^{-5} sq. ft. per sec., the critical Reynolds number is obtained when $dV = 2{,}000 \times 10^{-5}$ or $DV = 0.24$. Thus for a pipe 1 in. in diameter the critical velocity would be only 0.24 ft. per sec. Or if the velocity were 2.4 ft. per sec., the critical diameter would be 0.1 in. Either velocities or pipe diameters as small as these values are seldom encountered in engineering problems, though they may be met with in certain scientific or laboratory instruments. Hence, for such fluids as water and air, practically all of the cases with which the engineer has to deal are in the turbulent-flow region. But if the fluid were an oil with a kinematic viscosity one hundred times that of water, for example, it is seen that laminar flow would usually prevail.

93. General Equations for Frictional Resistance.—The following treatment applies to either laminar or turbulent flow and to any shape of cross section. Consider the forces on a stream of constant cross section A and length L, and assume that in the length L of a horizontal pipe there is a pressure difference $p_1 - p_2$. This pressure difference acting on the two ends will produce an axial force of $(p_1 - p_2)A$. For equilibrium in steady flow, this will be balanced by the shear at the wall, which will be $PL\tau_0$, where P is the perimeter and τ_0 the shear force per unit area. Equating these two forces,

$$(p_1 - p_2)A = PL\tau_0$$

or

$$(p_1 - p_2) = \frac{L}{m}\tau_0 \tag{120}$$

where $m = \dfrac{A}{P}$. (121)

For a smooth conduit, where wall roughness may be neglected

for the present, it may be assumed that the fluid shear at the wall is some function of m, ρ, μ, and V and that it may be expressed as

$$\tau_0 = Km^x \rho^y \mu^z V^n, \tag{122}$$

where K is a dimensionless number but is not necessarily a constant. Substituting dimensional values M, L, T for mass, length, and time in the preceding expression,

$$ML^{-1}T^{-2} = L^x(ML^{-3})^y(ML^{-1}T^{-1})^z(LT^{-1})^n,$$

where L denotes a linear unit, while it represents a length of channel in the rest of the discussion. Since the dimensions on the two sides of the equation must be alike,

For M,	$1 = y + z.$
For L,	$-1 = x - 3y - z + n.$
For T,	$-2 = -z - n.$

The solution in terms of n is $x = n - 2$, $y = n - 1$, $z = 2 - n$. Employing these values of the exponents in Eq. (122), the result is

$$\tau_0 = Km^{n-2}\rho^{n-1}\mu^{2-n}V^n. \tag{123}$$

This may also be written in the equivalent form

$$\tau_0 = K\left(\frac{mV\rho}{\mu}\right)^{n-2}\rho V^2 = 2KR^{n-2}\rho\frac{V^2}{2}, \tag{124}$$

where $mV\rho/\mu$ is a Reynolds number, since m is a linear dimension of significant importance. By definition, the friction coefficient is

$$c_f = 2KR^{n-2} = \frac{2K}{R^{2-n}} \tag{125}$$

and is an abstract number, since both K and R are dimensionless. Employing this *friction-drag* coefficient, from Eq. (124),

$$\tau_0 = c_f\rho\frac{V^2}{2}. \tag{126}$$

From Eqs. (120) and (124) we obtain

$$p_1 - p_2 = \frac{2K}{R^{2-n}}\frac{L}{m}\rho\frac{V^2}{2} = c_f\frac{L}{m}\rho\frac{V^2}{2}.$$

Dividing this last expression by $w = g\rho$,

$$h_f = \frac{p_1 - p_2}{w} = c_f\frac{L}{m}\frac{V^2}{2g}. \tag{127}$$

94. Pipes of Circular Cross Section.—For a pipe of circular cross section,

$$m = \frac{A}{P} = \frac{\pi r^2}{2\pi r} = \frac{r}{2} = \frac{d}{4}.$$

Since R for circular pipes is based upon d, it is seen that for noncircular sections $4m$ should be used in computing Reynolds number in order that they may be comparable.

When Eq. (127) is applied to a circular pipe, it becomes

$$h_f = f\frac{L}{d}\frac{V^2}{2g}, \tag{128}$$

where

$$f = 4c_f = \frac{8K}{R^{2-n}} \tag{129}$$

Like c_f, the friction factor f is dimensionless and is also some function of Reynolds number R. Since L/d is a ratio of two linear dimensions, it is also an abstract number. It is immaterial what units are employed for L and d, as long as they are both in the same dimensions. Equation (128) expresses the fact that the friction loss in a given pipe is a pure number times the velocity head.

Applying Eq. (120) to a circular pipe, $p_1 - p_2 = \tau_0 2L/r_0$, where r_0 is the radius to the pipe wall. Likewise, for any concentric cylindrical body of fluid of smaller diameter than the pipe, $p_1 - p_2 = \tau 2L/r$, where r is the radius to *any* point. From this it follows that the unit shear at any radius r is

$$\tau = \frac{\tau_0}{r_0}r, \tag{130}$$

or the unit shear is zero at the center of the pipe and increases directly with the radius to a maximum value τ_0 at the wall. This may also be seen from the fact that the end forces on the cylinder vary as the square of the radius, while the perimeter varies as the first power only. Thus the unit shear must also vary directly as the radius in order to maintain the balance.

From Eqs. (126) and (129) we obtain

$$\tau_0 = \frac{f}{4}\rho\frac{V^2}{2} = \frac{f}{4}w\frac{V^2}{2g}. \tag{131}$$

By the aid of this equation, τ_0 may be computed for any experimentally determined value of f.

Dimensional analysis gives us the proper form for an equation but does not yield a numerical result, since it is not concerned with abstract numerical factors. Hence it shows, as in Eq. (123), that whatever be the value of the exponent of V, the exponents of all the other quantities involved are then determined. It also shows that Eq. (128) is a rational expression for pipe friction. But the numerical values of such quantities as K, n, and f must be determined by other means.

95. Laminar Flow in Circular Pipes.—From Art. 5 it may be seen that for laminar flow $\tau = \mu \frac{du}{dy}$, where u is the value of the velocity at a distance y from the wall. Since $y = r_0 - r$, it is also seen that $\tau = -\mu \frac{du}{dr}$, or, in other words, the minus sign indicates that u decreases as r increases. The coefficient of laminar viscosity μ is a constant for any particular fluid at a constant temperature, and therefore, if the shear varies from zero at the center of the pipe to a maximum at the wall, it follows that the velocity profile must have a zero slope at the center and have a continuously steeper velocity gradient as the wall is approached.

In order to determine the velocity profile, the expression for τ in laminar flow will be inserted in the general relatior

$$p_1 - p_2 = \frac{\tau 2L}{r},$$

so that we have

$$p_1 - p_2 = -\mu \frac{du}{dr} 2 \frac{L}{r}.$$

From this

$$du = -\frac{p_1 - p_2}{2\mu L} r \, dr.$$

Integrating and determining the constant of integration by the fact that $u = U_{max}$ when $r = 0$, we obtain

$$u = U_{max} - \frac{p_1 - p_2}{4\mu L} r^2 \tag{132}$$

From this equation it is seen that the velocity profile is a parabola, as shown in Fig. 135.

Also, from the fact that the velocity at the wall is zero or $u = 0$, when $r = r_0$, the value of the velocity at the center is

$$U_{max} = \frac{p_1 - p_2}{4\mu L} r_0^2 = \frac{p_1 - p_2}{16\mu L} d^2. \tag{133}$$

Equation (132) may be multiplied by a differential area, and the product integrated over the area of the pipe in order to find the rate of discharge; and, as in previous cases, it will be found that the rate of discharge is proportional to a solid bounded by the velocity profile. In this case the solid is a paraboloid of

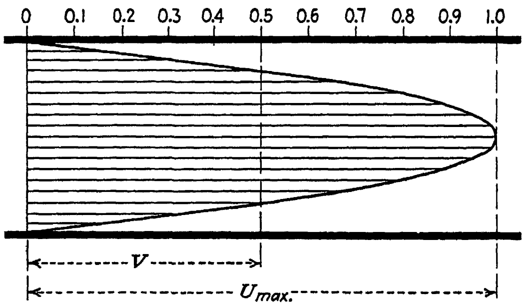


FIG. 135.—Velocity profile in laminar flow.

height U_{max}. But the mean height of a paraboloid is one-half the maximum height, and hence the average velocity is $V = 0.5U_{max}$. Thus

$$V = \frac{p_1 - p_2}{32\mu L} d^2. \tag{134}$$

From this last equation, and noting that $w = g\rho$ and that $\mu/\rho = \nu$, the loss of head is given by

$$h_f = \frac{p_1 - p_2}{w} = 32\frac{\mu}{w}\frac{L}{d^2}V = 32\nu\frac{L}{gd^2}V. \tag{135}$$

From this it is seen that in laminar flow the loss of head is proportional to the first power of the velocity. That is, for laminar flow $n = 1$. This is verified by experiment, as shown in Fig. 134. The striking feature of this equation is that it involves no empirical coefficients or experimental factors of any kind, except for the physical properties of the fluid such as viscosity and

specific weight. From it one might also conclude that the friction is independent of the roughness of the pipe wall. That this is true is borne out by experiment.

But dimensional analysis shows us in Eq. (128) that the friction loss may also be expressed by $h_f = f\left(\frac{L}{d}\right)\frac{V^2}{2g}$. Equating this to (135) and solving for the friction factor f, we obtain

$$f = \frac{64\mu}{dV\rho} = \frac{64}{R}. \tag{136}$$

Hence, we may then use either Eq. (135) or the equation

$$h_f = \frac{64}{R}\frac{L}{d}\frac{V^2}{2g}. \tag{137}$$

Laminar flow is dealt with rather fully, not merely because it is of practical importance in problems involving fluids of high viscosity but because it permits of a simple and completely rational analysis, and this, in turn, is of some assistance to us in the study of turbulent flow.

96. Entrance Conditions in Laminar Flow.—In the case of a pipe leading from a reservoir, if the entrance is properly rounded so as to avoid any contraction of the entering stream, all particles of fluid will start to flow down the pipe with the same velocity, except for a very thin film in contact with the walls. With this slight exception, all particles move with the same velocity V. The velocity profile is then a straight line across the pipe, and the kinetic energy of the stream per unit rate of discharge is $V^2/2g$. But as flow continues along the pipe, the outer particles are retarded, and the central portion accelerated, so that the velocity profile changes with distance, as shown in Fig. 136. The complete parabolic profile is an equilibrium condition and is theoretically reached only at infinite distance. Practically this condition may be assumed to be attained in a distance $x_1 = 0.13Rd$.[1] Thus for a value of R of 2,000, the length of pipe necessary for this transition is 260 diameters.

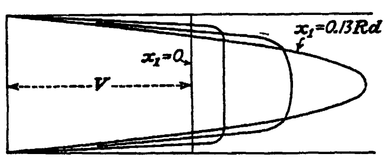


FIG. 136.—Velocity profiles along the pipe for laminar flow.

[1] PRANDTL-TIETJENS, "Applied Hydro- and Aeromechanics," p. 22, McGraw-Hill Book Company, Inc., 1934.

As shown in Art. 48, the kinetic energy of a stream with a parabolic velocity profile is $2V^2/2g$, where V is the mean velocity. Thus there is a continuous acceleration in this transition distance, as the kinetic energy gradually increases from the initial value of $V^2/2g$ to the final value of $2V^2/2g$.

In addition to the decrease in pressure due to pipe friction, there is a further drop in pressure due to the kinetic energy which the stream acquires, and thus this additional drop increases from $V^2/2g$ at the entrance to the pipe to the value $2V^2/2g$ where the velocity profile becomes parabolic.

EXAMPLES

179. An oil with a kinematic viscosity of 0.00015 sq. ft. per sec. flows through a pipe line which is 6 in. in diameter. What is the critical velocity? *Ans.* 0.6 ft. per sec.

180. An oil with a kinematic viscosity of 0.005 sq. ft. per sec. flows through a pipe 3 in. in diameter with a velocity of 10 ft. per sec. Is the flow laminar or turbulent?

181. Hydrogen at atmospheric pressure and a temperature of 50°F. has a kinematic viscosity of 0.0011 sq. ft. per sec. What would be its critical velocity in a pipe 2 in. in diameter? *Ans.* 13.2 ft. per sec.

182. Gasoline with a kinematic viscosity of 0.000008 sq. ft. per sec. is pumped through a pipe line 4 in. in diameter. What is its critical velocity? *Ans.* 0.048 ft. per sec.

183. Steam with a specific weight of 0.25 lb. per cu. ft. flows with a velocity of 100 ft. per sec. through a circular pipe. The value of f was found to be 0.016. What is the shear per unit area? *Ans.* 0.155 lb. per sq. ft.

184. If the oil in Prob. 180 weighs 58 lb. per cu. ft., what will be the friction drop in pressure in a distance of 3,000 ft.? *Ans.* 962 lb. per sq. in.

185. In Prob. 180 what will be the distance from the entrance before the approximate attainment of equilibrium? *Ans.* 16.25 ft.

97. Turbulent Flow in Circular Pipes.—Viscosity in laminar flow may be explained, especially in the case of gases, by the aid of the kinetic theory. This is that all gases are composed of molecules which continually move at random in all directions at high velocities, a continuous interchange of momentum being the result of their collisions with each other. The average distance traveled between collisions is called the mean free path.

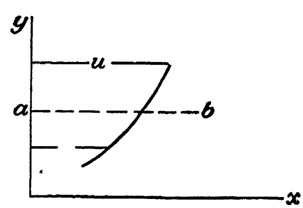


FIG. 137.

Consider a body of fluid moving to the right in Fig. 137 and with a velocity u which increases with y, as shown. Owing to the molecular motion, molecules will cross the line ab, even in laminar flow, and thereby transport momentum. Since, on the average, the velocities of the molecules in the slower moving fluid below the line will be less than those of the faster moving fluid above, the result is that the molecules which cross from below tend to slow up the faster moving fluid. Likewise the molecules which cross the line ab from above tend to accelerate the slower moving fluid below. The result is the production of a shear along the surface through ab the value of which is given in Art. 5 as $\tau = \mu\, du/dy$. The viscosity coefficient is a property of the fluid and depends upon the weight and number of molecules that pass over through a unit area in unit time. It is a constant in a given flow, but the shear force varies as the velocity gradient du/dy.

In the modern conception of turbulent flow, a similar mechanism is assumed. The molecules are considered to be replaced by minute but finite masses; the molecular velocity is replaced by the turbulent fluctuations of velocity; and, instead of the mean free path, there is a mixing length, which is the average distance traveled perpendicular to the main flow during which the small mass loses its excess of velocity and accommodates itself to its new surroundings. Hence, by analogy the shear along the plane through ab due to the turbulence may be expressed as $\epsilon \dfrac{du}{dy}$. But, unlike μ, the turbulent, or mechanical, viscosity coefficient ϵ is not a constant for a given fluid. Its value depends upon the turbulence of the flow and also varies according to the position in the stream. Its magnitude is said to range from 0 to 20,000 μ. However, its numerical value is of small importance, as it is the physical concept conveyed by it that chiefly concerns us. Since the laminar viscosity acts even in turbulent flow, the total shear is

$$\tau = \mu \frac{du}{dy} + \epsilon \frac{du}{dy}. \tag{138}$$

Turbulence vanishes at the wall, and hence ϵ is there zero. Therefore, the shear near the wall is due to the laminar viscosity alone, but it may be large because of the very steep velocity gradient in that region. In the central portion of the pipe the

laminar viscosity effect becomes negligibly small at high Reynolds numbers because of the flatness of the velocity profile. But even though du/dy is there comparatively small, the turbulent friction may be important because of the large value of ϵ.

The second term of Eq. (138) may be expressed in another form, which is also instructive. If at a given point the mean temporal value of the velocity is u, there may be considered as superimposed upon it velocity fluctuations u' and v' in the x and y directions, respectively. That is, u and u' are parallel to the axis of the pipe, and v' is perpendicular to it. Since u' and v' are fluctuations on both sides of a constant mean value, it follows that the temporal mean value of each one is zero. But the temporal mean value of the product $\overline{u'v'}$ is not zero, owing to a certain correlation between the two. Physically this correlation is that particles from a fast-moving fluid move into a slower moving fluid, and vice versa, so that on the average when a particle moves from a region of lower to one of higher velocity its contribution to the momentum is proportional to $-u'$, while in the reverse direction its momentum contribution is proportional to $+u'$. In the former case, if the mass that travels into the new region is proportional to $+\rho v'$, in the latter it must be proportional to $-\rho v'$. Thus, mathematically there is found the association of $-u'$ with $+v'$ and $+u'$ with $-v'$. Consequently, the net unit shearing force due to turbulence is represented by $-\rho\overline{u'v'}$. The total unit shear may now be represented as

$$\tau = \mu \frac{du}{dy} - \rho\overline{u'v'}, \tag{139}$$

where the last term is to be added arithmetically. Since v' must be zero at the wall, it follows from this equation that only laminar friction is possible near the wall. As in the previous case, this equation cannot be directly applied but is chiefly useful in the explanation that it offers as to the mechanism of turbulent friction.

In order to develop a relation that can be applied and that must therefore be expressed in terms of measurable quantities, Prandtl assumed a mixing length l and by a chain of reasoning, which we shall not take time for, derived the equation

$$\tau = \tau_0\left(1 - \frac{y}{r_0}\right) = \mu \frac{du}{dy} + \rho l^2\left(\frac{du}{dy}\right)^2. \tag{140}$$

From any experiment where the pipe friction is measured, the value of τ_0 can be calculated by Eq. (131); and if the velocity profile is also determined in the same case, the value of du/dy may be found for different radii, and finally the mixing length l may be evaluated. If the latter proves to be some definite function of the distance from the pipe wall, then by the aid of this function we can calculate the shear in all other cases. It has been found experimentally that l/r_0 always has the same value for a given value of y/r_0 whether the pipe be smooth or rough. Also, l/r_0 is zero at the wall, where there can be no fluctuations perpendicular to the axis, and increases in a perfectly definite way as the distance from the pipe wall increases.

Theodor von Kármán has developed an equation from which l may also be determined from the velocity profile alone. It is

$$l = K\frac{du/dy}{d^2u/dy^2}, \tag{141}$$

in which K has been found to be a universal constant with a value of about 0.40.

By these means it has been found possible to develop theoretical equations for the velocity profile and the friction coefficient in turbulent flow. Some of the recent results will be here given without any attempt to reproduce the derivations.

Except for a thin boundary layer near the wall, von Kármán has shown that

$$\frac{U_{max} - u}{\sqrt{\tau_0/\rho}} = \phi\left(\frac{r}{r_0}\right) \tag{142}$$

is a universal function. That is, the left-hand side of the equation always plots as one single curve regardless of the velocity, pipe size, or roughness. For the function of the right-hand side, he has derived

$$\phi\left(\frac{r}{r_0}\right) = -\frac{1}{K}\left[\log_e\left(1 - \sqrt{\frac{r}{r_0}}\right) + \sqrt{\frac{r}{r_0}}\right]. \tag{143}$$

Prandtl has recently published the equation

$$\phi\left(\frac{r}{r_0}\right) = 5.75 \log_{10}\left(\frac{r_0}{r_0 - r}\right). \tag{144}$$

While these two do not give identical results, they are very close together and are both in good agreement with the experimentally determined velocity profiles. Both of them give a profile with a sharp point at the center of the pipe, whereas the curve must be rounded there, since the tangent must approach zero as r approaches zero. But with this slight exception the two equations are valid up to $r = 0.99r_0$.

If it is possible to calculate the velocity profile, it may then be possible to compute the shearing stress and consequently the friction coefficient. In 1930 von Kármán derived a theoretical formula for the relation between f and R for smooth pipes. The derivation is too lengthy to present here, but it is based upon the principle of similarity. The equation is

$$\frac{1}{\sqrt{f}} = -0.8 + 2.00 \log_{10} (R\sqrt{f}). \tag{145}$$

This equation is valid for smooth pipes for all values of R from the critical value up to $R = \infty$, where $f = 0$. It is theoretically correct and agrees with experimental data. Its principal disadvantage is that it is not practicable to solve it directly for f for a given value of R. The procedure is to solve for R for various values of f and plot a curve for use. Such a curve will be found in Fig. 141.

The value of f for rough pipes will be greater than that given by Eq. (145), and so far no satisfactory theoretical equation that is of universal application has been developed. Further discussion of this will be found in Art. 103.[1]

EXAMPLES

186. Tests made on a certain pipe showed that when $V = 10$ ft. per sec., $f = 0.015$. The fluid used was water at 60°F. at which temperature the specific weight = 62.37 lb. per cu. ft. Find the unit shear at the wall and at a distance from the center of 0.5 the pipe radius.

Ans. 0.364 lb. per sq. ft., 0.182 lb. per sq. ft.

187. The absolute viscosity of water at 60°F. is 0.0113 poise, or 0.0000236 lb. sec. per sq. ft. If at the distance from the center of the pipe in Prob. 186 of $r = 0.5r_0$ the value of du/dy in a pipe 12 in. in diameter is found from the

[1] For details of this entire treatment of turbulent flow see Th. von Kármán, "Turbulence and Skin Friction," *J. Aeronautical Sciences*, vol. 1, no. 1, January, 1934, first published in 1930 in *Nachr. Ges. Wiss. Göttingen*, Heft 1, 58–76; "Some Aspects of the Turbulence Problem," *Mech. Eng.*, vol. 57, no. 7, July, 1935. PRANDTL-TIETJENS, *op. cit.*, page 195, Chap. III.

velocity profile to be 5.33 sec.$^{-1}$, find the laminar shear and the turbulent shear. *Ans.* 0.000126 lb. per sq. ft., 0.1819 lb. per sq. ft.

188. What is the mixing length l for the case in Prob. 187, and what is the ratio l/r_0? *Ans.* 0.0675 ft. or 0.69 in., 0.115.

189. For a pipe for which $f = 0.015$ when the maximum velocity in the center is 10 ft. per sec., compute the velocity at $r = 0.5r_0$ by Eq. (143). *Ans.* 9.435 ft. per sec.

190. Compute the result called for in Prob. 189 by the aid of Eq. (144). *Ans.* 9.25 ft. per sec.

191. For $f = 0.02$, compute the value of Reynolds number for a smooth pipe. *Ans.* 60,800.

98. Velocity Profile in Turbulent Flow.—In Fig. 138 are shown velocity profiles for both rough and smooth pipes, as computed by use of the equations given in the preceding article. As com-

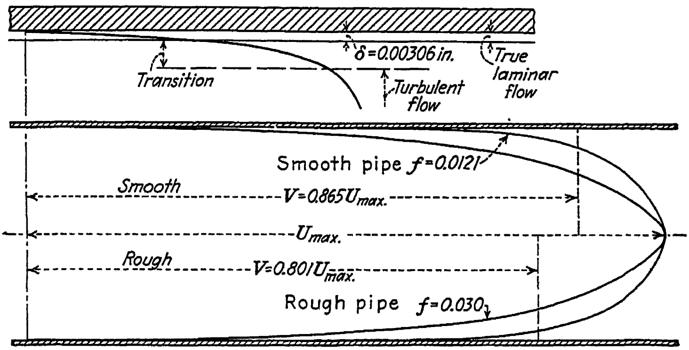


FIG. 138.—Velocity profiles in turbulent flow.

pared with laminar flow, it is seen that these curves are flatter in the central portion and steeper at the walls. The higher the Reynolds number the flatter this curve becomes. It is also flatter for a pipe with smooth than for one with rough walls.

One point of practical value is the ratio between the mean velocity and the maximum velocity. Nikuradse has obtained the equation

$$U_{max} - V = 4.07\sqrt{\frac{\tau_0}{\rho}},$$

and from this may be derived

$$\frac{V}{U_{max}} = \frac{1}{1 + 1.435\sqrt{f}}. \tag{146}$$

From this may be seen that the rougher the pipe and consequently the larger the value of f the lower the ratio. Also, since for a smooth pipe it may be seen from Eq. (129) that f decreases as R increases, the larger the value of R the higher the value of this ratio. The practical utility of this factor is that if the velocity in the center of the pipe alone is observed, the mean velocity may be computed from it.

99. Laminar Boundary Layer.—It has been seen that next to the wall of a smooth pipe there can be no turbulent fluctuation, since motion normal to the wall is suppressed. Also, the velocity of the fluid in contact with the wall is zero. The thickness of the laminar boundary layer may be estimated in the following manner.

Assuming that the velocity gradient near the wall may be represented by a straight line, then from the definition $\tau = \mu \dfrac{du}{dy}$ we obtain

$$\frac{\tau_0}{\rho} = \frac{\mu}{\rho}\frac{V'}{\delta}$$

where V' is the velocity at the outer edge of the boundary layer, and δ is the thickness. By employing the various relations already derived, the foregoing may be transformed as follows:

$$\delta = \frac{V'\nu}{\frac{\tau_0}{\rho}} = \frac{V'\nu}{\frac{fV^2}{8}} = 8\frac{V'}{V}\frac{d\nu}{f\,dV} = 8\frac{V'}{V}\frac{d}{fR}. \qquad (147)$$

This equation shows that the thickness of the laminar boundary layer is independent of the size of the pipe, since d appears also in R, but that it decreases with increasing roughness or velocity and increases with increasing kinematic viscosity. If the ratio of V' to V, the mean velocity, were known, the foregoing would serve to evaluate the thickness.

Actually, there is no sharp line of demarcation between the laminar flow and the turbulent flow, but one gradually merges into the other so that there can be said to be a region in which will be found a transition. From investigations of von Kármán the velocity at the edge of the true laminar layer is

$$V' = 8\sqrt{\frac{\tau_0}{\rho}}, \qquad (148)$$

while the velocity at the point where normal turbulent flow is fully developed is about $14\sqrt{\tau_0/\rho}$.

Inserting the value of V' from Eq. (148) in Eq. (147), the thickness of the true laminar layer is about

$$\delta = \frac{8\nu}{\sqrt{\dfrac{\tau_0}{\rho}}}. \tag{149}$$

Within the transition region, the velocity gradient may no longer be assumed to be a straight line, so its thickness cannot be determined quite so readily. But, by methods that will not be given here, it has been found that fully developed turbulence will be encountered when the distance from the wall is $30\nu/\sqrt{\tau_0/\rho}$. The space between this and the value given by Eq. (149) is the transition zone.

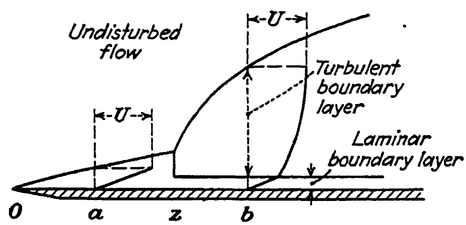


FIG. 139.—Development of boundary layer.

In the upper part of Fig. 138 is shown the laminar boundary layer with a greatly magnified scale.

100. Turbulent Boundary Layer.—In Fig. 139 is shown the case of a fluid with a uniform undisturbed velocity U flowing along a flat surface parallel to it. From the edge at o a laminar boundary layer forms with a thickness that increases as shown up to some "critical" value at z, at which point it abruptly drops to a smaller value, which then remains constant. It is this constant value with which we have been concerned in the preceding discussion.

Between o and z there is merely the laminar boundary layer separating the surface from the undisturbed flow, and at a the velocity profile is a straight line from zero up to the velocity U. But at z begins a turbulent boundary layer, which increases in thickness much more rapidly than did the original laminar layer,

and it continuously increases in thickness along the length of the plate. The velocity profile at b is a straight line in the laminar sublayer and a curved one in the turbulent boundary layer. It may be seen from the velocity profile that it would be difficult to determine the exact point that divides the turbulent layer from the undisturbed flow.

In the case of a single surface it is seen that there may be a turbulent boundary layer, with a laminar sublayer, but that outside the former there is an undisturbed velocity field. Such a condition may be found in the case of an airplane wing, an airship, a submerged submarine, or any other body of limited size in a large fluid field.

101. Entrance Conditions in Turbulent Flow.—If the surface in Fig. 139 be the wall of a pipe, the boundary layer from one wall will meet that from the other wall in a comparatively short distance from the entrance to the pipe, and then there will be no undisturbed field. Instead, there will be merely a laminar boundary layer at the wall, while the rest of the cross section is filled with the turbulent flow.

In the case of laminar flow in a pipe it was seen that a very considerable distance must be traversed from the entrance before the normal velocity profile is even approximately attained. This is because a long distance is necessary before the laminar layer shown from o to z will increase in thickness enough to extend close to the center of the pipe. But where the flow in the pipe is turbulent, the critical point z is reached in a very short distance from the end of the pipe, and then the thickness of the turbulent boundary layer increases so rapidly that the normal velocity profile is obtained in a much shorter distance than for laminar flow. The indications are that the normal velocity profile is attained in turbulent flow in a distance of about 20 to 40 diameters from the entrance.

In addition to the decrease in pressure due to pipe friction, there is a further drop of pressure due to the kinetic energy acquired. At the very end of the pipe the velocity is practically uniform across a diameter except very close to the wall, and so the kinetic energy is practically $V^2/2g$; but as the normal velocity profile is attained, the kinetic energy becomes slightly greater because of the fact that the velocity is not uniform over the section. For the normal type of velocity profile, the excess of

the true kinetic energy over that computed from the square of the mean velocity is usually less than $0.1V^2/2g$.

102. Surface Roughness.—It has been stated that the friction is independent of the wall roughness in the case of laminar flow. But this is very far from being true in the case of turbulent flow, for in the latter the effect of roughness is of the utmost importance.

There is no such thing in reality as a perfectly smooth surface, for under a powerful microscope any surface will appear to be rough. However, in the case of fluid flow, if the heights of the projections of the surface are less than the thickness of the laminar boundary layer, the effect is the same as if the surface were mathematically smooth. Hence, actual pipes, especially those made of such materials as glass, lead, brass, etc., may have the same friction losses as would be obtained with an ideally smooth surface.

Unfortunately, there is no definite and rational numerical scale of roughness in general, nor is there any direct way of measuring the roughness of a given surface. At some future date, practicable methods may be devised, but at the present time our knowledge of the effect of roughness is being greatly aided by experiments made upon channels with an artificial roughness which has some geometrical regularity. For example, the interior of a pipe may have screw threads cut in it. By varying the pitch or the depth of the thread, the roughness may be varied in regular steps. One of the most important of the experimental researches with artificial roughness has been that of Nikuradse, who coated pipe walls with sand grains of different sizes. This made it possible to express the roughness by the numerical values of r_0/e, r_0 being the radius of the pipe, and e the diameter of the sand grains. His investigations covered a range of values of r_0/e of 15 to 507. Obviously, the maximum limit of roughness would be when the heights of the projections would be as great as the radius, that is, $r_0/e = 1$, while for a perfectly smooth pipe e would be zero, and therefore $r_0/e = \infty$.

However, it is known that the shapes and the spacing of the roughness projections also affect the friction, so that the preceding ratio is not the sole criterion. But so far it has not been possible to express these other factors in any simple and general numerical fashion. Also, it seems that the irregularity found with natural

roughness yields somewhat different results from these purely artificial cases. Nevertheless, the information obtained by such studies is very significant.

It is important to distinguish between *absolute* roughness and *relative* roughness. For example, the value of e determines the absolute roughness, while r_0/e determines the relative roughness. Thus, a given surface may produce a large relative roughness in a small pipe and a small relative roughness in a large pipe. Since the variation in absolute roughness is usually much less than the possible range of diameters, it follows that small pipes are usually relatively rough while very large ones are comparatively smooth. As usually employed, the words rough and smooth refer to the relative roughness, as determined by r_0/e, since the pipe friction is a function of this ratio.

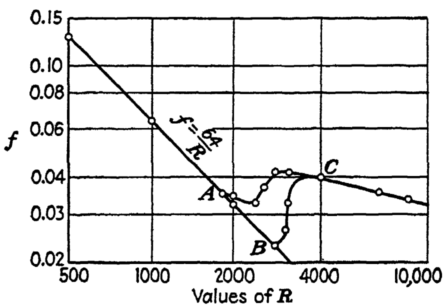


FIG. 140.

103. Values of f vs. R.—It has been shown that the friction factor f is a function of Reynolds number. The relation between these two in the neighborhood of the critical Reynolds number is shown in Fig. 140, where the points A, B, C correspond to the same letters, respectively, in Fig. 134.

In general, however, above the critical Reynolds number, f is a function not only of R but also of some roughness factor, such as r_0/e, for example. This is shown in Fig. 141. The lowest curve shows the minimum values of f such as are to be found with smooth pipes only, while other curves are shown for rough pipes up to about the maximum that would ever be found in any normal practice. The four curves designated by the r_0/e values were determined by Nikuradse with artificial roughness produced by sand grains. It will be observed that for $r_0/e = 507$ the curve is identical with that for smooth pipes until $R = 100{,}000$. This

is because for values of R below that the thickness of the laminar boundary layer is greater than the diameter of the sand grains. But as the thickness of the laminar boundary layer diminishes with increasing Reynolds number, a point is reached at which the sand grains pierce through the laminar layer, and then the action is that of a rough pipe. For a still rougher pipe, for example, with $r_0/e = 60$, the transformation from smooth to rough takes place at a much lower value of R, and the ultimate value of f is also higher.

It will be observed also that for rough pipes at high Reynolds numbers the value of f is a constant and independent of R, or, in other words, it is a function of the roughness alone. But if the roughness is produced by a waviness of the walls instead of by irregular projections, the curves for f would be higher than that for smooth pipe but would be approximately parallel to it, as shown by a small portion of one such curve.

In Fig. 141 there are also two curves reproduced from the work of Beale and Docksie, who made careful investigation in the neighborhood of the critical region with pipes of natural roughness.[1] It will be noticed that, though their curves are horizontal at large Reynolds numbers, as are the Nikuradse curves, they are quite different from them in shape in the critical region. This discrepancy may be due to the difference between regular artificial and irregular natural roughness but is more probably due to differences in entrance conditions. This simply illustrates the complexity of the problem in dealing with roughness.

The value of f for rough pipes for the Nikuradse scale of roughness is given by the equation

$$\frac{1}{\sqrt{f}} = 2.00 \log_{10} \frac{r_0}{e} + 1.74. \tag{150}$$

It is seen that this equation does not directly involve R. Thus for a given size of pipe, that is for a fixed value of r_0, the value of f will depend upon that of e alone, and the result will be a series of horizontal lines in Fig. 141, at least for large Reynolds numbers.

But if e is constant while r_0 varies, then f will be a function of r_0, and hence of Reynolds number. In this way the dash curves shown in Fig. 141 were obtained. Hence, the solid lines indicate

[1] BEALE, E. S. L., and DOCKSIE, P. "Flow in Pipes in Critical Region," *J. Inst. Petroleum Tech.*, vol. 18, no. 105, p. 607, July, 1932.

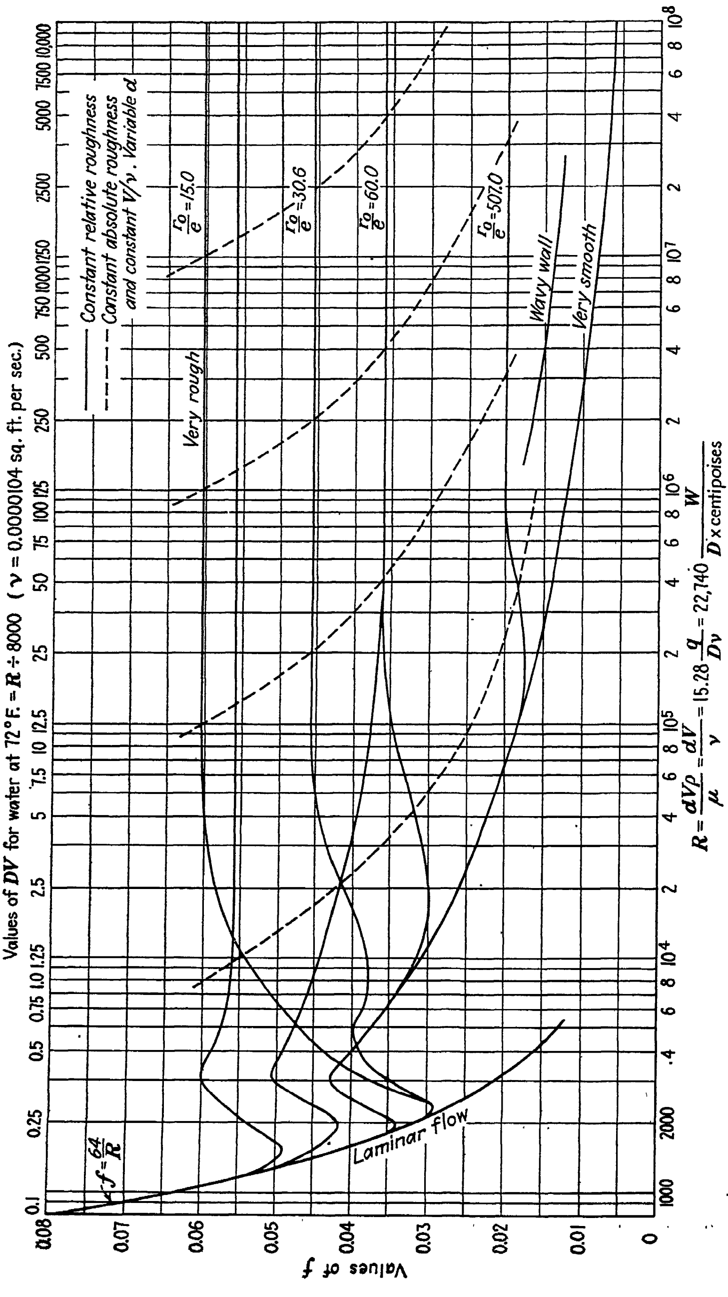


FIG. 141.—Values of friction factor.

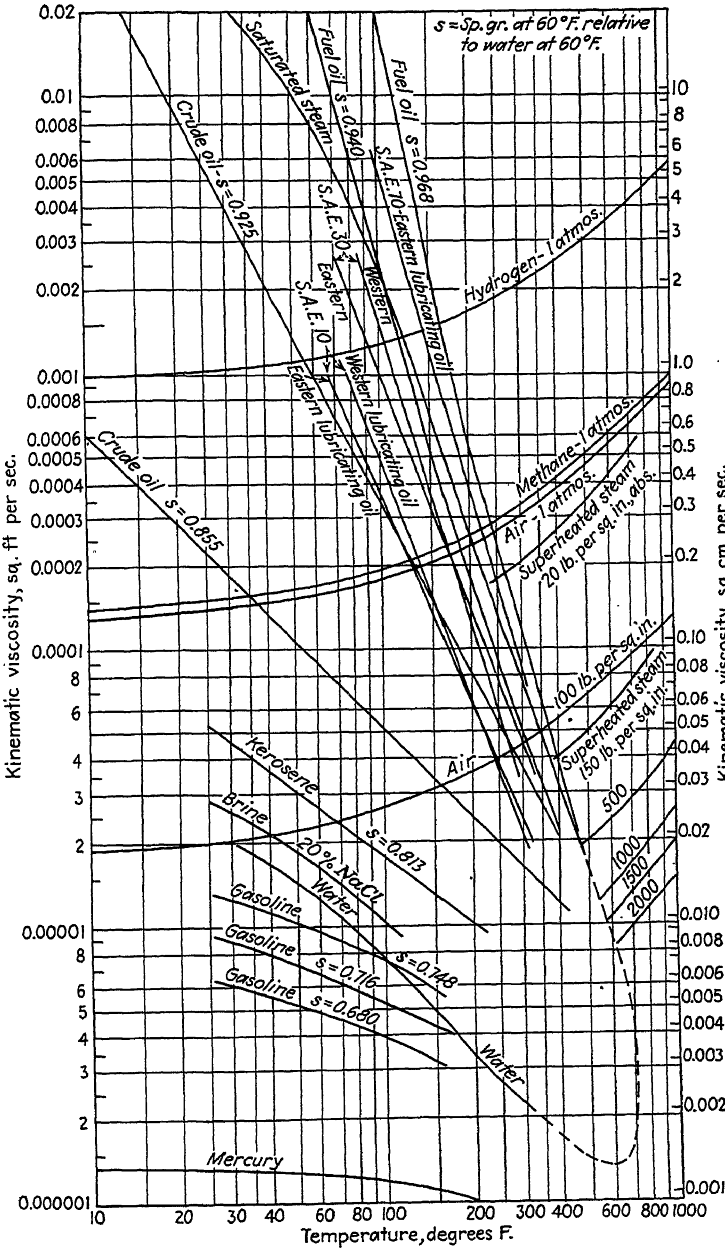


FIG. 142.—Values of kinematic viscosity.

curves of constant relative roughness, while the dash lines indicate curves of constant absolute roughness. The former are independent of Reynolds number, so long as it is large; the latter are not. If experiments are made upon a single pipe with any fluid flowing at different velocities or with different fluids of different kinematic viscosities, the values of f will all lie along one of these solid lines or one parallel to them. But if Reynolds number is varied by changing the diameter only, then the values of f will lie along one of the dash curves or one parallel to them.

In a series of commercial pipes of different diameters, but of the same material, the value of e is not necessarily constant, as in the case just considered. Instead, it may be true that the surface of the smaller sizes has a greater absolute smoothness, that is with smaller values of e, than the larger sizes. Hence, the curves for a series of commercial pipes of different sizes but of the same type may be found to lie in between these two cases previously discussed. Hence, it is obvious that the exact determination of f for a given case is still partly empirical and that much must depend upon the good judgment and experience of the engineer in estimating the effect of roughness.

EXAMPLES

192. What is the ratio of the mean velocity to the maximum velocity in a smooth pipe when (a) $R = 10{,}000$, (b) $R = 10{,}000{,}000$?

Ans. (a) 0.798, (b) 0.888.

193. Air at a pressure of 1 atm. and a temperature of 100°F., flows through a pipe 18 in. in diameter with a velocity of 12 ft. per sec. What is the velocity at the edge of the laminar boundary layer, and what is the thickness of the laminar layer? *Ans.* 4.55 ft. per sec., 0.0286 in.

194. What is the loss of head in feet of the fluid flowing in 100 ft. of pipe with a diameter of 6 in. if the velocity is 10 ft. per sec. and the pipe roughness is such that $r_0/e = 507$? *Ans.* 6.21 ft.

195. In the case of a smooth pipe 12 in. in diameter and a mean velocity of 5 ft. per sec., find the value of f for (a) water at 32°F., (b) water at 300°F., (c) air at 100 lb. per sq. in. abs. and 80°F., (d) crude oil ($s = 0.855$) at 100°F., (e) saturated steam at 400°F. *Ans.* 0.0150, 0.0100, 0.0155, 0.0180, 0.0160.

196. Find the value of f for a pipe 6 in. in diameter if $e = 0.01$ in.

Ans. 0.0222.

104. Empirical Determination of f.—An empirical method for the determination of the friction factor, based upon the study of some 4000 test points by Kemler,[1] has been devised by

[1] KEMLER, E., "A Study of the Data of the Flow of Fluids in Pipes," *Trans. A.S.M.E.*, HYD-55-2, 1933.

Pigott.[1] Although the method is strictly empirical and not altogether rational, it does involve the fundamental principles presented in the preceding article, and the results obtained are in close agreement with the numerous test data.

The procedure is to select a curve number by the aid of Table VII, which involves both the type of pipe surface and the diameter. Then the value of f may be obtained by using this curve number in Fig. 143. For example, consider a clean steel pipe 4 in. in diameter. This is class B in Table VII, and for 4-in. diameter curve 5 is specified. Turning next to Fig. 143, the value of f is to be read for curve 5 for whatever value of Reynolds number may be involved.

TABLE VII.—SELECTED LOCATION OF f BY ROUGHNESS RELATION

Curve	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
A	0.35 up	....	....	....	....	....	....	....	...	...	...	...	..	0.125	..	..	..	0.0625
B	72	48 to 66	14 to 42	6 to 12	4 to 5	2 to 3	1½	1 to 1¼	¾	½	⅜	¼	⅛					
C	....	....	30	10 to 24	6 to 8	3 to 5	2½	1½ to 2	1¼	1	¾	½	..	⅜	..	¼	..	⅛
D	....	....	48 to 96	20 to 48	12 to 16	5 to 10	3 to 4	2 to 2½	1½	1¼	1							
E	....	....	96	42 to 96	24 to 36	10 to 20	6 to 8	4 to 5	3									
F	....	....	220	84 to 204	48 to 72	20 to 42	16 to 18	10 to 14	8	5	4	3						

A = drawn tubing, brass, tin, lead, glass, diameter, inches; B = clean steel, wrought iron, diameter, inches; C = clean, galvanized, diameter, inches; D = best cast iron, cement, light-riveted sheet ducts, diameter, inches; E = average cast iron, rough-formed concrete, diameter, inches; F = first-class brick, heavy riveted, spiral riveted, diameter, inches. In drawn tubing, actual inside diameter is given; in pipe, nominal size of standard weight is given.

EXAMPLES

197. Find the value of f for a spiral riveted pipe 12 in. in diameter with water at 72°F. and a velocity of 8⅓ ft. per sec. *Ans.* 0.019.

198. Water at 72°F. flows through a clean galvanized-iron pipe at the rate of 0.015 cu. ft. per sec. The nominal diameter is ¼ in., and the actual diameter 0.36 in. Find the loss of head per 100 ft. of length. (Use nominal diameter in Table VII but use actual diameter for computing velocity and Reynolds number.) *Ans.* 1,120 ft.

[1] PIGOTT, R. J. S., "The Flow of Fluids in Closed Conduits," *Mech. Eng.*, vol. 55, No. 8, p. 497, August, 1933.

105. Empirical Exponential Formulas.—From Eq. (123) it is seen that the friction loss may also be represented by an equation of the form

$$h_f = f' \frac{L}{d^{3-n}} V^n, \tag{151}$$

where f' is no longer an abstract number, as f was found to be.

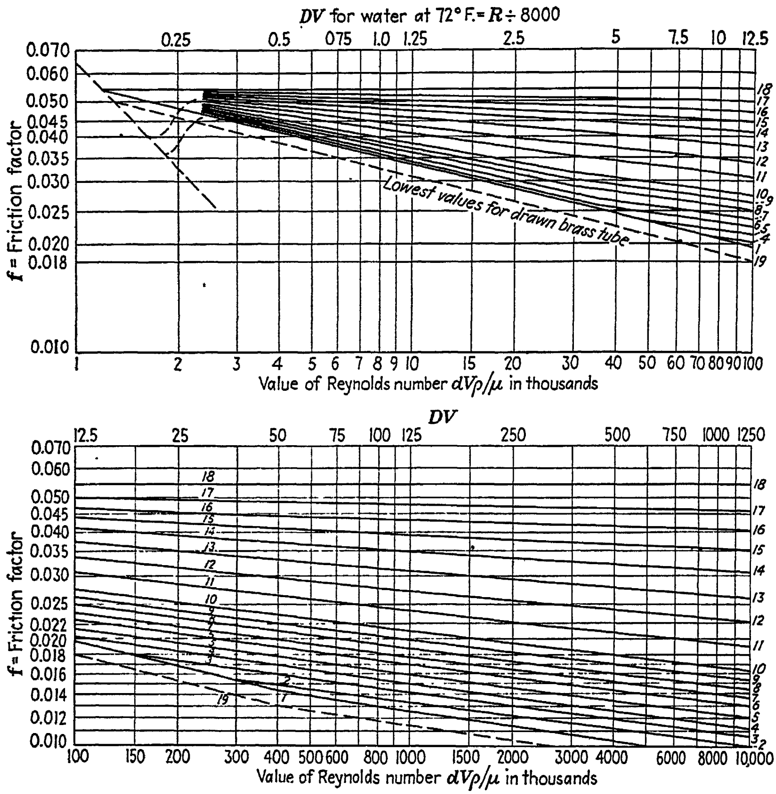


FIG. 143.—Friction factors for pipes.

The value of n would be 1 for laminar flow, in which case the formula would reduce to Eq. (135); and for large Reynolds numbers for rough pipe the value of n would be 2, in which case the formula would reduce to Eq. (128). But for other cases the value of n usually lies between 1.75 and 2.0.

This equation may be solved for velocity so as to obtain

$$V = \left(\frac{1}{f'}\right)^{\frac{1}{n}} d^{\frac{3-n}{n}} \left(\frac{h_f}{L}\right)^{\frac{1}{n}}$$

$$= C' m^{\frac{3-n}{n}} S^{\frac{1}{n}}, \qquad (152)$$

where m is used instead of d in order that it may be applied to all forms of section, and S is used for the ratio h_f/L.

Various empirical formulas have been based upon these fundamental equations, but, because of the fact that relative roughness usually varies with the diameter even though the surface is of the same type, they usually take the forms

$$h_f = f' \frac{L}{d^x} V^n \qquad \text{or} \qquad V = C' m^y S^{\frac{1}{n}},$$

where the exponents x and y are not related to n in the precise manner shown by Eqs. (151) and (152). The authors of these various formulas have then selected average values of f', x, and n or C', y, and n for different types of pipe or wall surfaces.[1] Since it may be seen from Fig. 143 that n varies with Reynolds number, it is obvious that the exponential formulas cannot be used to cover all cases without having the exponents n and x or y vary as well as f' or C'. This fact would appear to nullify any advantage that these formulas might appear to have. Therefore these empirical exponential formulas apply to a limited range only and cannot safely be used for all conditions. However, for this limited range they may sometimes be very convenient and can be accurate if the various factors are chosen to fit the conditions.

106. Noncircular Sections.—The discussion of the flow in closed pipes is largely confined to circular cross sections, because that shape is the most common and also because of the fact that most of the data have been obtained by experiments with this shape only. If the shape of the cross section is not circular, then the diameter in all the preceding formulas must be replaced with

[1] For example, Schoder has given $h_f = 0.00038LV^{1.86}/d^{1.25}$ for cast- and wrought-iron pipes in fair condition and $h_f = 0.0005LV^{1.95}/d^{1.25}$ for rough and riveted pipes. The well-known Hazen-Williams formula is

$$V = C' m^{0.63} S^{0.54},$$

where C' is selected from a table for different types of surfaces.

$4m$, where m = cross-sectional area divided by wetted perimeter. If the shape of the cross section is such that the velocity adjacent to each portion of the surface is substantially the same, the evidence seems to be that for turbulent flow the friction factor is the same for a given value of m as it is for a circular pipe for which $d = 4m$.

If the flow is laminar, the problem is more complicated, and the friction is invariably higher than it would be in the corresponding circular form. Experiments have shown friction losses of as much as 50 per cent higher for some noncircular sections with laminar flow.

107. Minor Losses.—Whenever the velocity of a flowing stream is altered either in direction or in magnitude, such alteration sets up additional eddy currents and thus creates a loss of energy in excess of the usual pipe friction. The magnitude of this loss is proportional to the abruptness of the velocity change. It differs from normal pipe friction in that the first effect is to increase the kinetic energy of turbulence, represented by B^2 in Eq. (28), and kinetic energy of this form cannot be converted back into some useful form such as pressure but instead is gradually transformed into heat downstream. Though the disturbing factor is usually confined to a very short length of path, the effects may not disappear for a considerable distance.

It is customary to refer to such losses as minor losses because in a pipe line of considerable length the pipe friction itself may be so large that the value of these other losses may be relatively insignificant.

Thus assuming a pipe line whose length is 10,000 diameters and with a friction factor of 0.02, the value of fL/d is 200. Suppose that the loss of head due to these other causes is $1.2V^2/2g$. It may be seen that this is only 0.6 per cent of the pipe friction and is a negligible quantity in view of the fact that the value of f may not be known within 10 per cent, for example.

On the other hand, if the length is 500 diameters, the value of fL/d is 10, so that it does make an appreciable difference in the computed result whether the $1.2V^2/2g$ is omitted or not. Even in this case, however, the value of the minor losses is no more than a possible variation in the friction factor.

But if the length of the pipe is only 10 diameters, the pipe friction becomes only $0.2V^2/2g$, and it is seen that the so-called

minor losses now become the most significant factor. Thus the importance of such items is seen to be a matter of relative proportions.

The most common sources of minor losses are described in the remainder of this chapter. Such losses may be represented in either one of two ways. They may be expressed as $kV^2/2g$ where k must be determined for each case, or they may be represented as being equivalent to a certain length of straight pipe. However, instead of a definite length of pipe L this is given as equal to a certain number of pipe diameters L/d. This value is then added to the L/d corresponding to the actual length of the pipe.

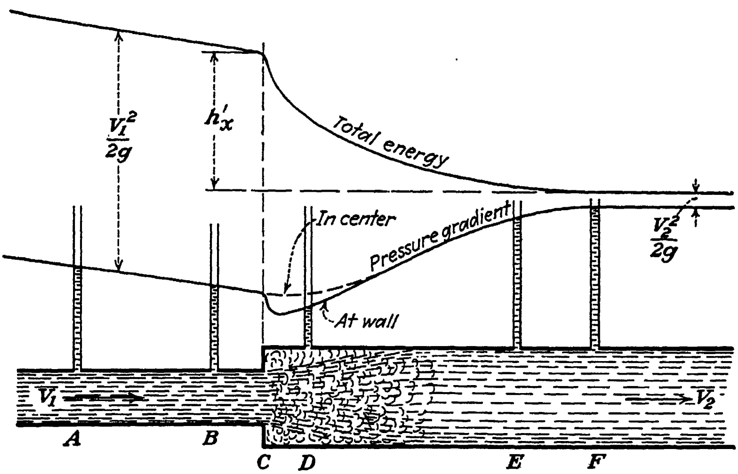


FIG. 144.—Loss of head caused by sudden enlargement.

108. Loss Due to Sudden Enlargement.—The conditions at a section of sudden enlargement are shown in Fig. 144. There is a rise in pressure because of the decrease in velocity, but this rise is not so great as it would be if it were not for the loss in energy. There is a state of excessive turbulence from C to F beyond which the flow is normal. The drop in pressure just beyond section C, which was measured by a piezometer not shown in the illustration, is due to the fact that the pressures at the wall of the pipe are in this case less than those in the center of the pipe.[1]

[1] Figures 144 and 150 are plotted to scale from observations made by the author with the same velocities in both cases.

An expression for this loss of head can be derived as follows. In Fig. 145 assume that the pressure at (2) in the ideal case without friction is p_0. Then in this ideal case

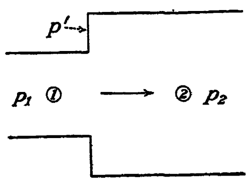


FIG. 145.

$$\frac{p_0}{w} = \frac{p_1}{w} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g}.$$

If in the actual case the pressure at (2) is p_2 while the average pressure on the annular ring is p', then equating the resultant force on the body of fluid between (1) and (2) to the time rate of change of momentum between (1) and (2), we obtain

$$p_1A_1 + p'(A_2 - A_1) - p_2A_2 = \frac{w}{g}(A_2V_2^2 - A_1V_1^2).$$

From this

$$\frac{p_2}{w} = \frac{A_1}{A_2}\frac{p_1}{w} + \frac{A_2 - A_1}{A_2}\frac{p'}{w} + \frac{A_1}{A_2}\frac{V_1^2}{g} - \frac{V_2^2}{g}.$$

The loss of head is $h'_x = (p_0 - p_2)/w$, and noting that

$$A_1V_1 = A_2V_2$$

and that $A_1V_1^2 = A_1V_1V_1 = A_2V_2V_1$, we obtain

$$h_x' = \frac{(V_1 - V_2)^2}{2g} + \left(1 - \frac{A_1}{A_2}\right)\left(\frac{p_1}{w} - \frac{p'}{w}\right). \tag{153}$$

It is usually assumed that $p' = p_1$, in which case the loss of head due to sudden enlargement is

$$h_x' = \frac{(V_1 - V_2)^2}{2g}, \tag{154}$$

which may also be written

$$h_x' = \left(\frac{A_2}{A_1} - 1\right)^2\frac{V_2^2}{2g} = \left(1 - \frac{A_1}{A_2}\right)^2\frac{V_1^2}{2g}. \tag{155}$$

Although it is possible that under some conditions p' will equal p_1, it is also possible for it to be either more or less than that value, in which case the loss of head will be either less or more than that given by Eq. (154). The exact value of p' will depend upon the manner in which the fluid eddies around in the corner

adjacent to this annular ring. However, the present information is that the deviation from Eq. (154) is quite small and of negligible importance.

109. Loss of Head at Discharge.—The energy loss at discharge from the end of a pipe line discharging into a large body of the same fluid may be regarded as a limiting case of sudden enlargement, where the final velocity V_2 is zero and the area A_2 is infinite. Thus from Eq. (154) or (155) the discharge loss is

$$h_d' = \frac{V^2}{2g}, \tag{156}$$

where V is the velocity of the fluid leaving the end of the pipe. But, as in the preceding case, the accuracy of this result depends upon the equality of p' and p_1, and, if certain eddies or induced currents in the surrounding fluid alter this pressure relation, the discharge loss may be slightly different from the value given by Eq. (156). But the difference, if any, is very small and can be disregarded. This loss could be materially reduced by placing a diverging mouthpiece on the discharge end of the pipe.

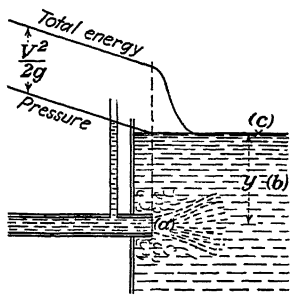


Fig. 146.—Conditions with submerged discharge.

This same equation can be independently derived by writing the energy equation between a point (a) at the end of the pipe in Fig. 146 and any point (b) where the fluid is at rest. Assuming the pressure on the stream at (a) to be equal to the depth y, then

$$H_a = y + \frac{V^2}{2g} \qquad \text{and} \qquad H_b = y + 0.$$

Consequently the loss of head between (a) and (b) is

$$h_d' = y + \frac{V^2}{2g} - y = \frac{V^2}{2g}$$

as in Eq. (156). There seem to be some indications that induced eddy currents may permit the pressure at (a) to be a trifle less than that due to the depth y, in which case the discharge loss will be slightly less than $V^2/2g$, but the difference, if any, is probably negligible, as previously stated.

Experience in teaching has shown that the beginner needs a word of caution regarding the use of discharge loss. If the energy equation is written between some point upstream and point (*a*) in Fig. 146, then there is no discharge loss to be considered, because the velocity head is included as a part of the total head at (*a*). But if the equation is written between some point upstream and point (*b*) or point (*c*) where there is no velocity, then the discharge loss does enter into the equation. The final result will be the same in either case, because in the general equation $H_1 = H_2 + h_f$ it is immaterial whether the $V^2/2g$ be included in the H_2 or in the h_f terms. It is merely necessary to be consistent and not include it in both places.

Again, in such a case as water discharging into air, as shown in Fig. 147 one may write the energy equation between (1) and (*B*)

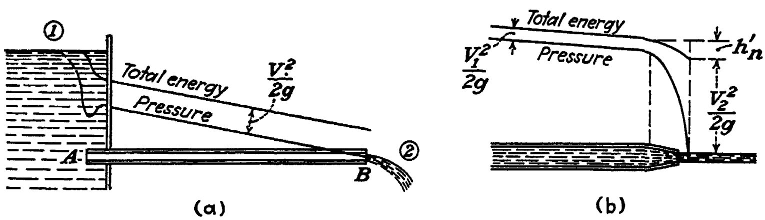


FIG. 147.—Conditions at discharge into air.

where there is a velocity V or between (1) and (2) where the velocity is disregarded. In the former case there is no discharge loss, but in the latter there is.

In Fig. 147 (*b*) is shown a nozzle on the end of a pipe. This is similar to the preceding case, except that the velocity of the jet is higher than the velocity within the pipe. The drop in the energy gradient marked h_n' is the friction loss within the nozzle itself and is similar to the friction within the pipe. It has nothing to do with discharge loss.

110. Loss of Head at Entrance.—Referring to Fig. 148, it may be seen that, as fluid from the reservoir enters the pipe, the streamlines tend to converge, much as though this were a jet issuing from a sharp-edged orifice, so that at B a maximum velocity and a minimum pressure are found. It may be supposed that at B the central stream is surrounded by fluid which is in a state of turbulence but has very little forward motion. Between B and C the fluid is in a very disturbed condition as the stream

expands and the velocity decreases, while the pressure rises. From C to D the flow is normal.

It is seen that the loss of energy at entrance is not confined to the section at A but is distributed along the length AC, a distance of several diameters. The increased turbulence and vortex motion in this portion of the pipe cause the friction loss to be much greater than in a corresponding length where the flow is normal, as is shown by the drop of the total energy line. Of this total loss a portion h_p' would be due to the normal pipe friction. Hence the difference between this and the total, or h_e', is the true value of the extra loss caused at entrance.

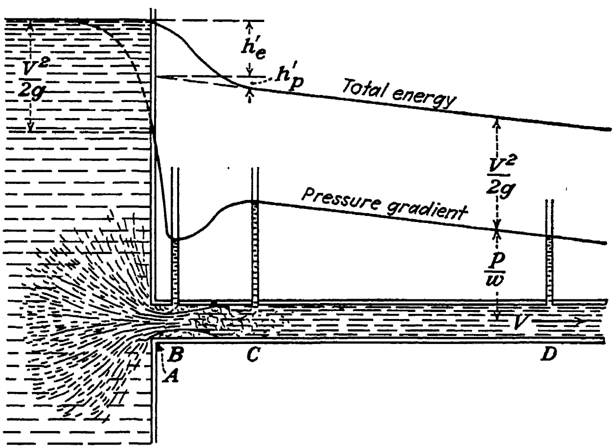


FIG. 148.—Conditions at entrance.

It is apparent that at section C the pressure is less than the static pressure due to the head in the reservoir by an amount equal to the velocity head acquired plus the total friction loss in AC.

It is seen that the entrance loss is caused principally by the sudden enlargement of the stream after it passes section B. Hence if we knew the effective area of the stream at B, this energy loss could be computed by the use of Eq. (155). Thus if A is the area of the pipe and c_cA is the effective area of the stream at B, the loss due to sudden enlargement is $\left(\frac{1}{c_c} - 1\right)^2\frac{V^2}{2g}$, where V is the normal velocity in the pipe. For example, if c_c be assumed

to be 0.59 for a square-edged orifice, then the entrance loss for such a case as shown in Fig. 148 would be $0.484V^2/2g$. If, however, the pipe were to project within the reservoir in the manner of a Borda tube, the streamlines would tend to converge more, and the value of c_c would be less. Assuming it to be 0.5 for such a case, the equation gives $\left(\frac{1}{0.5} - 1\right)^2\frac{V^2}{2g} = \frac{V^2}{2g}$.

Owing to the practical difficulty of measuring the contraction of the streams for different types of entrance conditions, it is more common to compute entrance loss from velocity coefficients determined by experiments on short tubes such as are shown in Fig. 91. In such cases the velocity coefficient is the same as the discharge coefficient, and the loss of head in the tube is determined by the expression $\left(\frac{1}{c_v^2} - 1\right)\frac{V^2}{2g}$. Thus, if for Fig. 91 (b) the coefficient be taken as 0.82, the resulting loss of head is $0.487V^2/2g$. Since this latter figure must include the normal pipe friction in the length of tube, it is seen that the agreement with the previous value is very good.

It should be emphasized that for the former case the contraction coefficient is for the entrance to the tube or pipe only, while for the latter the velocity coefficient is for the complete tube, and the contraction coefficient for the complete tube is unity.

It is seen that the greatest entrance loss is to be found with a pipe that projects within the reservoir. However, the contraction coefficient is never quite so small as 0.5, so that the loss is never quite so great as $V^2/2g$. The actual value is also much affected by the relation between the thickness of the pipe and its diameter. As the thickness of the pipe wall becomes greater, the contraction of the entering stream becomes less, so that the loss decreases. Thus there may be a considerable range of values of the coefficient for this case. A very thick wall would approach the flush type of entrance in its characteristics. The most desirable type of entrance is the rounded or bell-mouthed entrance so formed as to eliminate all contraction. This reduces the loss to nothing more than normal friction. In fact, a very little rounding is effective.

Entrance losses may also be materially reduced by a very short conical mouthpiece on the end of the pipe. It has been found that the most efficient proportions of such a device is with a

central angle (that is, angle of vertex of cone) between 30 and 60 deg., and an area ratio of only 1:2 is necessary. This gives an entrance loss of $0.18V^2/2g$.[1]

If the loss of head at entrance is expressed as

$$h_e' = k_e\frac{V^2}{2g}, \tag{157}$$

values of k_e may be obtained as follows:

Bell-mouthed entrance	$k_e = 0.04$
Conical entrance	$= 0.18$
Nonprojecting pipe	$= 0.47$ to 0.56
Projecting pipe	$= 0.62$ to 0.93

For practical purposes where the entrance losses are quite secondary in importance, the values given in Fig. 149 may be used.

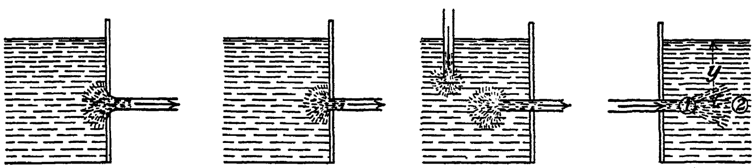


FIG. 149.—Entrance and discharge losses.

111. Loss Due to Sudden Contraction.—The phenomena attending the sudden contraction of a stream are shown in Fig. 150, which shows a marked drop in pressure due both to the increase in velocity and to the loss of energy in turbulence.[2] It is noted that in the corner upstream at section C there is a rise in pressure due to the fact that the streamlines are here curving so that the centrifugal action causes the pressure at the pipe wall to be greater than in the center of the stream, for which the pressure variation from B to C is indicated by the dotted line.

From C to E the conditions are similar to those described for entrance. The loss of head may be represented by

$$h_c' = k_c\frac{V_2^2}{2g} \tag{158}$$

where k_c has the values given in Table VIII.

[1] SEELY, F. B., Univ. Illinois *Eng. Exp. Sta. Bull.* 96.

[2] Figures 144 and 150 are plotted to scale.

TABLE VIII.

d_2/d_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
k_c	0.45	0.42	0.39	0.36	0.33	0.28	0.22	0.15	0.06

It may be observed from the scale of the drawings that the loss of head due to sudden enlargement is greater than that due to sudden contraction. This is an illustration of the well-known fact that the losses of energy accompanying a decrease in velocity are always greater than those accompanying an increase. This is

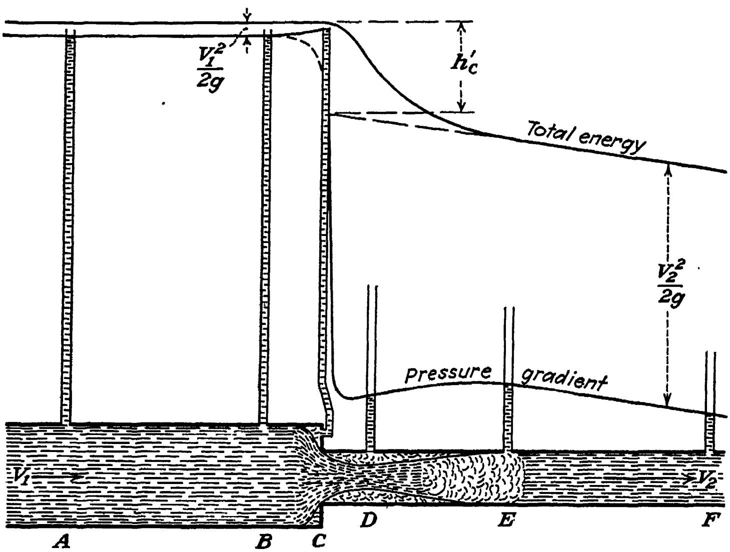


FIG. 150.—Loss of head caused by sudden contraction.

due to the less stable flow and the greater turbulence in the former case.

112. Loss for Gradual Contraction.—In order to reduce the foregoing losses, abrupt changes of cross section should be avoided. Inspection of Fig. 150 shows that the principal cause of loss is due to the contraction of the stream as it enters the smaller pipe. This may be prevented by changing from one diameter to the other by means of the smooth curve *AC* of Fig. 151 (*a*). With this construction there will be no additional loss in the length *CD*. Owing, however, to the higher average

velocity in the portion AC, there will be a slightly greater loss than in the same length of pipe of the original diameter. This loss is identical with that in a nozzle of the same form as the portion AC and is practically negligible. Ordinarily it will be about $0.04V_2^2/2g$.

For ease in manufacture the curved portion AC of Fig. 151 (*a*) may be replaced with a frustum of a cone without any appreciable increase in the loss. If the angle of the cone is too great, the conditions of sudden contraction will be approached, owing to the tendency of the streamlines to converge downstream from C. On the other hand, if the angle of the cone is too small, the distance AB will be unduly large, and, since the average velocity in

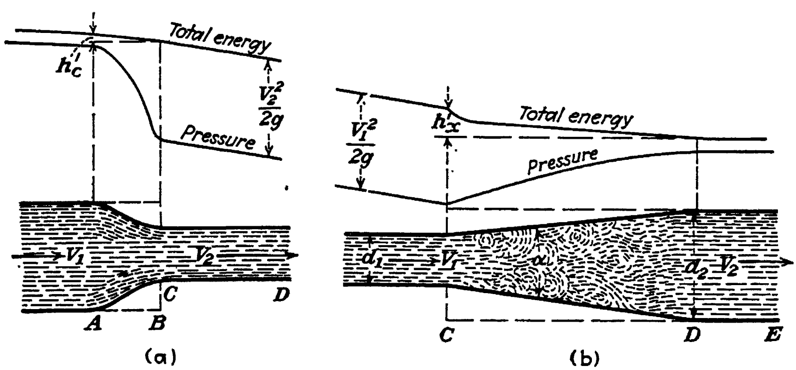


FIG. 151.—Gradual changes in area.

this conical section will be higher than that in the larger straight pipe, the friction loss in the reducer will be somewhat greater than in the length of straight pipe which it replaces. A total angle of from 20 to 40 deg. is probably about the proper amount of taper.

113. Loss for Gradual Enlargement in Diffuser.—A diffuser to join the smaller to the larger diameter may also be given a curved outline, or it may be a frustum of a cone, as in Fig. 151 (*b*). The loss of head will be some function of the angle of divergence and also of the ratio of the two areas, the length of the diffuser being determined by these two variables. Because of the great importance of this problem in such cases as draft tubes for turbines, diffuser passages for centrifugal pumps, and other practical applications, it seems desirable to devote some space to the consideration of the factors involved.

First, for a given angle α the loss in the diffuser must increase as the ratio d_2/d_1 increases owing to the greater length of CD. Next, with a given ratio of d_2/d_1 the length of the section CD and the loss within it will vary with the angle α.

In flow through a diffuser the total loss may be considered as made up of two factors. One is the ordinary pipe friction loss, which may be represented by

$$h_f = \int \frac{f}{d} \frac{V^2}{2g}(dL).$$

(Note that (dL) is a differential length of the cone, while d is the diameter at any section.) In order to integrate the foregoing it is necessary to express the variables f, d, and V as functions of L. For our present purpose it is sufficient, however, merely to note that the friction loss increases with the length of the cone. Hence for a given ratio of areas, the larger the angle of the cone the less its length and the less the pipe friction, which is indicated by the curve marked F in Fig. 152. However, in flow through a diffuser there is an additional turbulence loss set up by induced currents which produce a vortex motion over and above that which normally exists.[1] This additional turbulence loss will

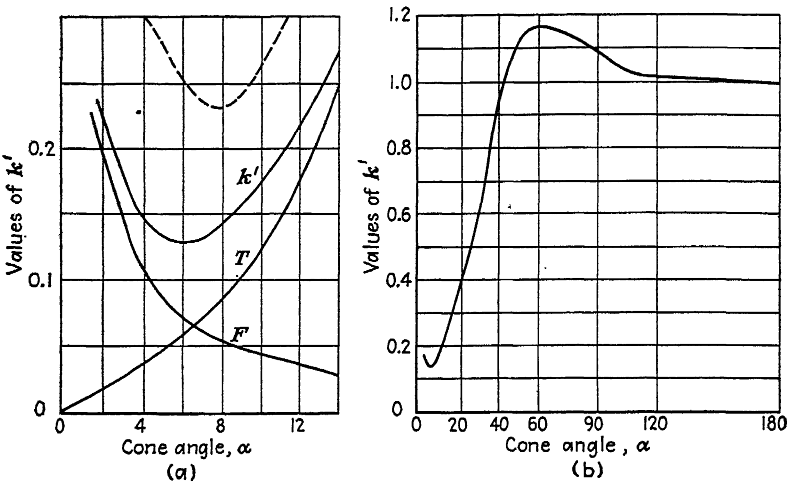


FIG. 152.—Losses in conical diffuser.

[1] NEDDEN, F. ZUR, "Induced Currents of Fluids," *Trans. A. S. C. E.*, vol. 80, p. 844, 1916.

naturally increase with the degree of divergence, as is indicated by the curve marked T in Fig. 152. The total loss in the diverging cone is then represented by the sum of these two losses, marked k'. This is seen to be a minimum at 6 deg. for the particular case chosen, which is for a very smooth surface. If the surface were rougher, the value of the friction F would be increased. This not only increases the value of k', which is now indicated by the dotted curve, but also shifts the angle for minimum loss to 8 deg. Thus the best angle of divergence increases with the roughness of the surface.

It has been found that this excess turbulence loss, produced by secondary or induced currents set up, is independent of the roughness of the surface of the cone. But it is a function of the viscosity of the fluid. The higher the viscosity the less the amount of eddying in the fluid and the less the loss. Decreasing the amount of this turbulence loss by using a liquid of higher viscosity than water tends to shift the angle of minimum loss to a higher value, as may be readily seen. But with a higher viscosity the normal pipe friction would also be greater, and this likewise calls for a greater angle. Thus the best angle of divergence increases with an increase of viscosity.

It has been seen that the loss due to a sudden enlargement is very nearly represented by $(V_1 - V_2)^2/2g$. The loss due to a gradual enlargement is expressed as

$$h' = k'\frac{(V_1 - V_2)^2}{2g} = k'\left(\frac{A_2}{A_1} - 1\right)^2\frac{V_2^2}{2g} = k'\left(1 - \frac{A_1}{A_2}\right)^2\frac{V_1^2}{2g}. \quad (159)$$

Values of k' as a function of the cone angle α are shown in Fig. 152 (b), for a wider range than appears in (a).[1] It is of interest to note that at an angle slightly above 40 deg. the loss is the same as that for a sudden enlargement, which is 180 deg., and that between these two the loss is greater than for a sudden enlargement, being a maximum at about 60 deg. This is because the induced currents set up are worse within this range.

[1] GIBSON, A. H., *Engineering* (*London*), Feb. 16, 1912. These values were based on area ratios of 1:9, 1:4, 1:2.25 and gave one curve up to an angle of about 30 deg. Beyond that the three ratios gave three curves which differed as much as about 18 per cent at 60 deg., where the turbulence was a predominating factor, and then drew together again as 180 deg. was approached. The curve here shown is a composite of those three.

EXAMPLE

199. For a diameter ratio of 1:2 and a velocity of 20 ft. per sec. in the smaller pipe, find the loss of head due to (*a*) sudden contraction, (*b*) sudden enlargement, (*c*) expansion in conical diffuser with total angle of 20 and 6 deg.

Ans. (*a*) 2.05 ft., (*b*) 3.50 ft., (*c*) 1.40 ft., 0.453 ft.

114. Loss in Pipe Fittings.—The loss of head in pipe fittings may also be expressed as $kV^2/2g$, where V is the velocity in a pipe of the nominal size of the fitting. Values of k are given in Table IX.[1]

TABLE IX

Fitting	k
Globe valve, wide open	10
Angle valve, wide open	5
Close return bend	2.2
Tee, through side outlet	1.8
Short-radius elbow	0.9
Medium-radius elbow	0.75
Long-radius elbow	0.60
45-deg. elbow	0.42
Gate valve, wide open	0.19

115. Loss in Pipe Bends.—In flow around a pipe bend the velocity profile is greatly distorted as a result of centrifugal action. This distortion produces disturbances in the flow not only in the bend itself but for a distance of about 100 diameters in the straight pipe immediately following. In fact, more than half of the energy loss produced by the curved bend takes place in this section of straight pipe. Because of this, the loss is not greatly dependent upon the angle subtended. Thus two bends placed together so as to turn the fluid through an angle of 180 deg. produce very little more friction than a single 90-deg. bend. However, if there is a short length of straight pipe between the two, the loss is greater.

The frictional resistance produced by a bend is some function of the ratio of the radius of curvature to the diameter of the pipe, but the relation is not a simple one. The loss is large for a very small value of this ratio and diminishes rapidly as the ratio increases, but, instead of diminishing continuously to zero as the

[1] Crane Co., "Flow of Fluid in Pipes and Heat Transmission," 1935. Experimental results determined by the Crane Co. Univ. Wis. and Univ. Texas.

ratio increases to infinity, there are a series of local maxima and minima.

The friction losses produced by a pipe bend may be expressed as equivalent to an additional length of straight pipe, values of which may be obtained by the aid of Table X. The ratio of the radius of curvature to the diameter of the pipe is given by R_c/d, the equivalent length of straight pipe which represents the resistance due to the curvature is L_1. But the greater the radius of curvature the longer the length of the pipe in the bend itself. Adding this actual length L to the equivalent length L_1 gives a length L_2 which is a measure of the total resistance produced by the pipe bend.

TABLE X.—EQUIVALENT LENGTHS FOR 90° BENDS

R_c/d	L_1/d	L/d	L_2/d
1	18	1.5	19.5
2	12	3	15
3	11	4.5	15.5
5	13	7.5	20.5
7	15	10	25
10	12	16	28
12.5	10	19	29
15	11	23	34
20	12	30	42
25	10	40	50
30	8	47	55

116. PROBLEMS

200. Prove that for a constant rate of discharge and a constant value of f the friction loss in a pipe varies inversely as the fifth power of the diameter.

201. Prove that $d = \sqrt[5]{fLq^2/39.7h_f}$.

202. A smooth pipe 12 in. in diameter and 300 ft. long has a flush entrance and a submerged discharge. The velocity is 10 ft. per sec. If the fluid is water at 72°F., what is the loss of head? (Use Fig. 141.) *Ans.* 7.92 ft.

203. Suppose that the fluid in Prob. 202 were oil with a kinematic viscosity of 0.001 sq. ft. per sec. and a specific gravity of 0.925; what would be the friction loss in feet of oil and in pounds per square inch?

Ans. 16.8 ft., 6.7 lb. per sq. in.

204. Suppose that the fluid in Prob. 202 were dry saturated steam at a pressure of 29.82 lb. per sq. in. abs., at which pressure its temperature is 250°F. and its specific volume 13.824 cu. ft. per lb. If the velocity were also 10 ft. per sec., what would be the energy loss in feet of steam and in pounds per square inch? *Ans.* 10.7 ft., 0.00538 lb. per sq. in.

205. Suppose that the velocity in Prob. 204 were 130 ft. per sec.; what would be the drop in pressure in pounds per square inch?

Ans. 0.67 lb. per sq. in.

206. A smooth pipe consists of 50 ft. of 8-in. pipe followed by 300 ft. of 24-in. pipe with an abrupt change of cross section at the junction. It has a flush entrance and a submerged discharge. If it carries water at 72°F. with a velocity of 18 ft. per sec. in the smaller pipe, what is the total frictional resistance? *Ans.* 11.01 ft.

207. In a 100-ft. length of 4-in. wrought-iron pipe (Fig. 143) there is one open globe valve, one medium-radius elbow, and one pipe bend with a radius of curvature of 40 in. (The length of the bend is not included in the 100 ft.) No entrance or discharge losses are involved. If the fluid is water at 72°F., and the velocity is 6 ft. per sec., what is the total frictional resistance? *Ans.* 9.68 ft.

208. A cement pipe 6 ft. in diameter carries water at a temperature of 47°F. with a velocity of 5 ft. per sec. What is the loss of head per 1,000 ft. of length? *Ans.* 0.840 ft.

209. If the pipe were 4 in. in diameter, but all other data the same as in Prob. 208, what would be the value of the head lost? *Ans.* 26.8 ft.

210. A heavy spiral riveted-steel pipe 6 ft. in diameter carries water at 47°F. with a velocity of 5 ft. per sec. What is the loss of head per 1,000 ft. of length? *Ans.* 1.23 ft.

CHAPTER IX

FLOW THROUGH PIPES

117. Pipe Line Discharging into Air.—To illustrate the method of solution for flow through a pipe line of uniform diameter discharging freely into the air, a numerical example is given. Referring to Fig. 153, let h = 260 ft., the diameter of the pipe = 10 in., and the length = 5,000 ft. Consider the entrance to be nonprojecting and assume the value of k_e to be 0.5 for this case. The value of f to be used will depend not only upon the type and

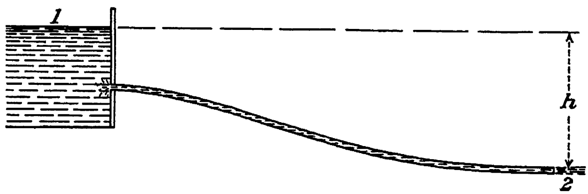


FIG. 153.

size of pipe but also upon the Reynolds number. Since the latter involves the velocity and is the unknown quantity in this problem, it is necessary to assume some reasonable value of f and compute the resulting velocity. If this differs materially from the value used in estimating the value of f, a new and corrected value of f may then be determined, and the solution repeated. Since the value of f does not change very much with a moderate variation in Reynolds number, this second solution will usually be sufficiently accurate. The greater the relative roughness of the pipe the less is the variation of f with R, and hence for rough pipes a second solution is often unnecessary.

In this particular case let us arbitrarily assume $f = 0.022$. Let (1) refer to a point at the surface of the liquid in the reservoir, while (2) refers to the stream issuing from the end of the pipe. Then $H_1 = 0 + 260 + 0$, $H_2 = 0 + 0 + V^2/2g$, and

$$h_f = \left(0.5 + 0.022 \times 5{,}000 \times \frac{12}{10}\right)\frac{V^2}{2g} = 132.5\frac{V^2}{2g}.$$

Inserting these values in the equation $H_1 = H_2 + h_f$,

$$260 = \frac{V^2}{2g} + 132.5\frac{V^2}{2g}$$

is obtained. Hence $V^2/2g = 260/133.5 = 1.95$.

$$V = 8.02\sqrt{1.95} = 11.2 \text{ ft. per sec.}$$

Therefore, $q = 0.545 \times 11.2 = 6.11$ cu. ft. per sec. It may be seen that with this length of pipe it would have made very little difference if the entrance loss and also the velocity head at (2) had been neglected altogether. This would have been equivalent to the assumption that

$$h = f\left(\frac{L}{d}\right)\frac{V^2}{2g}, \tag{160}$$

which is often used for long pipe lines, because in such cases the loss in pipe friction alone is approximately equal to h. But this approximate equation can be used only for long pipes of uniform diameter and when the velocity of the stream at (2) is no greater than in the pipe itself.

The velocity of 11.2 just computed may now be used in conjunction with the kinematic viscosity of the fluid to determine the Reynolds number and the corresponding value of f then obtained by Fig. 143. If this differs materially from the 0.022 previously assumed, the solution may be repeated as explained previously.

In case the pipe is made up of sections of different diameters, the procedure would be to write an expression for the loss of head in each length in terms of the dimensions applying to it, that is,

$$h_f = f_1\left(\frac{L_1}{d_1}\right)\frac{V_1^2}{2g} + f_2\left(\frac{L_2}{d_2}\right)\frac{V_2^2}{2g} + \text{etc.}$$

By the equation of continuity these might then all be placed in terms of one velocity head. The solution of such a problem is very similar to the example worked out in the next article.

By way of presenting some general laws it may be observed that for any pipe line, no matter how complex, the loss of head may be expressed as

$$h_f = y\frac{V^2}{2g}, \tag{161}$$

where y is a factor evaluated according to the various details of the pipe, and V is the velocity at any one section. That is, $y = \Sigma k + \Sigma fL/d$. From this,

$$V = \left(\frac{1}{\sqrt{y}}\right)\sqrt{2gh_f}.$$

If the general equation of energy is written between points (1) and (2) of Fig. 153, and V is now considered the velocity at (2), then

$$h - \frac{V^2}{2g} = y\frac{V^2}{2g},$$

from which

$$V = \left(\frac{1}{\sqrt{1+y}}\right)\sqrt{2gh}, \tag{162}$$

which is seen to be similar in form to the formula for the velocity of flow through an orifice. By introducing the area A,

$$q = AV = \left(\frac{A}{\sqrt{y}}\right)\sqrt{2gh_f} = K'\sqrt{h_f} \tag{163}$$

$$= \left(\frac{A}{\sqrt{1+y}}\right)\sqrt{2gh} = K\sqrt{h}, \tag{164}$$

where K' and K are factors representing the quantities indicated in the equations.

In case the pipe does not discharge into the air at (2), but the liquid there is under some pressure at that section, then h must be interpreted as the drop of the hydraulic gradient. Thus the foregoing relations are perfectly general.

Inspection of Eq. (160) shows that for a given head and length of pipe the velocity will vary somewhat with the diameter of the pipe. By Fig. 143, f decreases as d increases, and in Eq. (160) the ratio L/d also becomes smaller with larger diameters; therefore the entire coefficient of $V^2/2g$ becomes smaller as the size of the pipe increases. Hence for the same value of h, V will increase as the diameter of the pipe increases, and it may be shown that V varies as $d^{0.5 \text{ to } 0.6}$.

EXAMPLES

211. Suppose that in Fig. 153 the pipe projects into the reservoir at entrance and discharges freely into the air at (2), the size of the jet being equal to the diameter of the pipe. Assume $f = 0.02$ and $k_e = 0.9$. If $h = 40$ ft., $D = 12$ in., and $L = 50$ ft., compute the rate of discharge, considering all losses. *Ans.* 23.4 cu. ft. per sec.

212. In Prob. 211 if $L = 1{,}000$ ft., all other data remaining the same, compute the rate of discharge considering all losses. Compute the rate of discharge by the approximate method, neglecting minor factors.

Ans. 8.5 cu. ft. per sec.; 8.93 cu. ft. per sec.

118. Pipe Line with Nozzle.—Suppose in the example of the preceding article it is assumed that there is placed on the end of the pipe a nozzle which discharges a jet 2.5 in. in diameter and that the velocity coefficient of the nozzle is 0.95. There are two velocities to deal with instead of one, so let V_1 = velocity in the pipe and V_2 = velocity of the jet. By the equation of continuity, $A_1V_1 = A_2V_2$, and thus $V_2 = (10/2.5)^2V_1 = 16V_1$ and $V_2^2 = 256V_1^2$. The loss of head in the nozzle may be found by Eq. (57) to be $\left(\frac{1}{0.95^2} - 1\right)\frac{V_2^2}{2g} = 0.11\frac{V_2^2}{2g}$. If (2) now refers to the jet, while n refers to a point in the pipe near the nozzle, then

$$\begin{aligned} H_1 &= 0 + 260 + 0, \\ H_n &= p_n + 0 + \frac{V_1^2}{2g}, \\ H_2 &= 0 + 0 + \frac{V_2^2}{2g} = 256\frac{V_1^2}{2g}, \\ h_{f1-2} &= 132.5\frac{V_1^2}{2g} + 0.11\frac{V_2^2}{2g} = 160.6\frac{V_1^2}{2g}. \end{aligned}$$

Writing the equation now between (1) and (2),

$$260 = 256\frac{V_1^2}{2g} + 160.6\frac{V_1^2}{2g} = 416.6\frac{V_1^2}{2g}.$$

Thus

$$\frac{V_1^2}{2g} = \frac{260}{416.6} = 0.625,$$

and

$$V_1 = 8.02\sqrt{0.625} = 6.33 \text{ ft. per sec.}$$

It would have been equally easy to have substituted for V_1 in terms of V_2 and to have found $V_2^2/2g = 160$ and $V_2 = 101.6$ directly. With the procedure used, they can now be found by use of the equation of continuity. The rate of discharge is $q = 0.545 \times 6.33 = 0.034 \times 101.6 = 3.45$ cu. ft. per sec. This shows that the addition of the nozzle has reduced the discharge but has given a much higher jet velocity.

Writing the energy equation between (1) and n,

$$p_n + \frac{V_1^2}{2g} = 260 - 132.5\frac{V_1^2}{2g}.$$

Since the numerical value of $V_1^2/2g$ has been obtained, this may be reduced to

$$p_n = 260 - 133.5 \times 0.625 = 176.6 \text{ ft.}$$

which is the pressure head at the base of the nozzle. The solution may now be considered as completed, but, as a check, it may be noted that

$$H_n = 176.6 + 0.625 = 177.2$$

and

$$V_2 = 0.95 \times 8.02\sqrt{177.2} = 101.6 \text{ ft. per sec.}$$

EXAMPLE

213. A pipe line 400 ft. long and 6 in. in diameter discharges a 2-in. jet into the air at a point which is 200 ft. lower than the water surface at intake. The entrance to the pipe is a projecting one with $k_e = 0.9$, and the velocity coefficient of the nozzle is 0.97. Find the pressure head in the pipe at the base of the nozzle. What is the rate of the discharge? Assume $f = 0.02$. *Ans.* 165 ft., 2.20 cu. ft. per sec.

119. Submerged Discharge.—In case the discharge end of the pipe line is submerged, as shown in Fig. 154, the point (2) is most conveniently taken at the surface, where everything may be supposed to be known. Thus if $h = 100$ ft., the diameter $= 10$ in., and the length $= 4{,}000$ ft., then $H_1 = 100$, and $H_2 = 0$, taking the datum plane at the surface of the lower reservoir. Assuming at entrance, $k = 0.9$, and at discharge $= 1$, and $f = 0.022$, the total losses are given as

$$h_f = \left(1.9 + 0.022 \times 4{,}000 \times \frac{12}{10}\right)\frac{V^2}{2g} = 107.5\frac{V^2}{2g}.$$

Since $100 - 0 = 107.5V^2/2g$, therefore

$$\frac{V^2}{2g} = \frac{100}{107.5} = 0.93,$$

from which $V = 8.02\sqrt{0.93} = 7.72$ ft. per sec.

It may be observed that the solution is independent of the location of either end of the pipe, as the flow is a function only of the difference of the two levels. The result is also the same as if the pipe discharged freely into the air at a point whose elevation was 100 ft. lower than that of the surface at the intake end.

120. Size of Pipe for Given Discharge.—If there is given the loss of head in pipe friction only, or if the length is so great that all other losses are negligible, the size of pipe for a given rate of discharge can be determined as follows: From the continuity

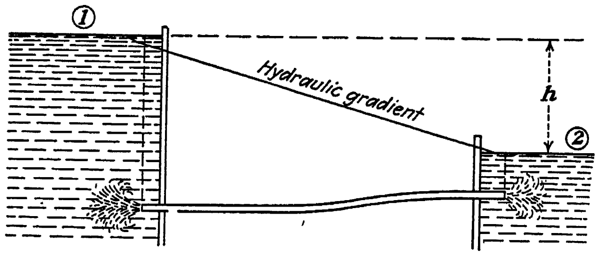


FIG. 154.

equation, $V = q/A = 4q/\pi d^2$, and therefore $V^2 = 1.62q^2/d^4$. Substituting this expression for V^2 in the equation for pipe friction, $h_f = f\frac{L}{d}\frac{V^2}{2g}$ and, rearranging, we obtain

$$d^5 = \frac{fL}{39.7h_f}q^2 \qquad \text{or} \qquad D^5 = \frac{6{,}270fL}{h_f}q^2. \qquad (165)$$

A value for f may be assumed more or less arbitrarily, and an approximate value of the pipe diameter computed by this equation. The velocity is then found for the given rate of discharge in this size of pipe. Having the diameter and the velocity, the Reynolds number can be obtained for any given fluid, and the corresponding value of f determined by the aid of Fig. 143. If this is materially different from the first value used, it may be necessary to repeat the computation, but any further calculation will rarely be necessary. In general, the pipe diameter so

obtained will not be a standard pipe size, and the final value will be selected by the aid of a table of standard sizes such as Table XIII (page 448).

As an illustration, suppose that it is desired to determine the size of standard wrought-iron pipe necessary to discharge 5 cu. ft. per sec. with a loss of head of 20 ft. in 1,200 ft. of length. Assuming $f = 0.025$, Eq. (165) will give $D = 11.85$ in. Since the corresponding area is 0.765 sq. ft., the velocity will be

$$V = \frac{5}{0.765} = 6.53 \text{ ft. per sec.}$$

If the fluid is water at a temperature of 60°F. at which the kinematic viscosity is 1.22×10^{-5} sq. ft. per sec., the value of Reynolds number is $11.85 \times 6.53 \times 10^5/(12 \times 1.22) = 528{,}000$. For this size of wrought-iron pipe, curve 4 is to be used in Fig. 143, and by it the value of f is found to be 0.016. Using this corrected value of f in Eq. (165), a corrected value of D is found to be 10.85 in.

While this new value of D would give a new value of Reynolds number, it is apparent that the change in the latter is less than might be caused by a variation in water temperature, and in either event the change in the resulting value of f would be small. In view of the fact that the values of pipe-friction factors found in any diagram are necessarily average values and may be in error in a specific case by as much as 5 per cent, excessive refinement in calculation is unwarranted. Also, in consulting the tables it may be seen that it will be necessary to use either a nominal 10-in. pipe with an actual diameter of 10.19 in. or a nominal 12-in. pipe with an actual diameter of 12.09 in. Which one of these two sizes should be specified would depend upon whether the capacity must be at least 5 cu. ft. per sec. or a slightly smaller discharge would be permissible. It would also depend upon whether a margin should be provided against a reduction in capacity in future years due to deposit of scale in the pipe. This has the effect of increasing the roughness, therefore raising the value of f, and it also reduces the cross-sectional area of the pipe.

If the circumstances are such that the velocity head in the pipe and the friction losses due to pipe fittings and other causes are not negligible in comparison with the pipe friction, the problem is more complicated than the preceding. If all such items

could be expressed in equivalent lengths of pipe, the solution would reduce to the case just described. But if there are friction losses which are expressed in the form $kV^2/2g$, the energy equation applied to such a case as is shown in Fig. 153 is

$$h = \left[\left(1 + \Sigma k + f\frac{L}{d}\right)\right]\frac{V^2}{2g}. \tag{166}$$

Substituting for V^2 in terms of q and d, this becomes

$$d^5 = \left[\left(1 + \Sigma k\right)d + fL\right]\frac{q^2}{39.7h}. \tag{167}$$

This equation must be solved by trial. Since the term involving d is usually small compared to the other, it may be omitted for the first trial, and the value of d so obtained then inserted on the right-hand side for a second trial. However, the solution is more readily obtained by trial in Eq. (166).

Thus, let $L = 200$ ft. of wrought-iron pipe, $h = 7.25$ ft., and $\Sigma k = 9$. It is required to determine the size of pipe necessary to deliver 0.5 cu. ft. of water per sec. Assume that $f = 0.025$ and try a 1¼-in. pipe for which Table XIII shows $D = 1.38$ in. and $A = 0.0104$ sq. ft. By Eq. (166)

$$\left(1 + 9 + 0.025\frac{200 \times 12}{1.38}\right)\frac{V^2}{2g} = 7.25,$$

from which $V^2/2g = 0.1355$ and $V = 2.95$. Then

$$q = 0.0104 \times 2.95 = 0.0307 \text{ cu. ft. per sec.}$$

The desired result is about sixteen times this value, but sixteen times the area will not be required, since the velocity will increase as the diameter of the pipe increases.

It would be reasonable to think that a pipe with an area about ten times the former value would be about right, but this size will be found to lie between a 4- and a 5-in. pipe. For the latter size and by the same procedure as before, $V = 4.62$ ft. per sec., and $q = 0.139 \times 4.62 = 0.642$ cu. ft. per sec. By comparison of the areas alone, it can be seen that the 4-in. pipe would be too small. The next step is to compute the Reynolds number and determine the corresponding f. In this case for water it will be found to be about 0.020, but even this will not suffice to make the 4-in. pipe adequate. Hence, the 5-in. pipe must be used.

The advantage of this method of solution is that one can make a close estimate of the size to use for the second or the third trial, but a third is seldom necessary. Also, one has a measure of the exact performance to be expected from the size finally selected. The method is seen to involve very little labor, especially in view of the fact that computations should be restricted to commercial sizes.

EXAMPLES

214. What must be the size of spiral riveted-steel pipe to discharge 80 cu. ft. of water per sec. with a pipe 5,000 ft. in length and with a pipe friction loss of 10 ft.?

215. A wrought-iron pipe line is to carry crude oil whose kinematic viscosity is 0.001 sq. ft. per sec. with a frictional resistance of 5 ft. per 1,000 ft. of length. If the flow is to be 2.0 cu. ft. per sec., what size pipe will be required?

216. If the oil in Prob. 215 were heated so as to reduce the kinematic viscosity to 0.0001 sq. ft. per sec., what would be the size pipe required?

121. Economic Size of Pipe.—Where the physical conditions fix the value of the head to be lost in pipe friction, as in the case of a pipe line discharging under gravity flow, the size of pipe for a given rate of discharge is to be determined as shown in Art. 120. But in case the pipe line is to deliver the fluid from a pump, the friction head may have any value whatever; while if it supplies water to a turbine, the head lost may be of any magnitude up to the total head available minus the velocity head in the pipe.[1] Since the friction head is not definitely fixed by physical conditions, its value must be established by other considerations.

If the rate of discharge is assumed to be constant, it is clear that the larger the pipe the less the velocity of flow and hence the less the value of the head lost. Since lost head means a waste of power in pumping any fluid, or a loss of power which might otherwise be developed by a power plant, the problem becomes one of determining the most economic value for this item.

The larger the pipe the more it costs, as is shown in Fig. 155. The values plotted, however, are a certain percentage of the total cost, being the annual fixed charges, which include such items as interest on the investment, depreciation, etc. In general, this curve is a discontinuous function, since the costs of

[1] Practically, the loss of head in a pipe leading to a turbine should be restricted to less than one-third the total head, as shown by Fig. 160.

different sizes of commercial pipe do not follow a single mathematical equation. This is particularly true if there is a change in the type of construction between one range of sizes and another. This may produce one or more abrupt steps in such a cost curve.

For each size of pipe the loss of head can be calculated, and this in turn fixes the amount of power lost in pipe friction. If this horsepower lost is then multiplied by the annual value of a horsepower, a curve showing the annual loss due to pipe friction can be plotted. The sum of these two items is the total annual cost of the pipe line. The diameter for which this total cost is a minimum is the most economica size.

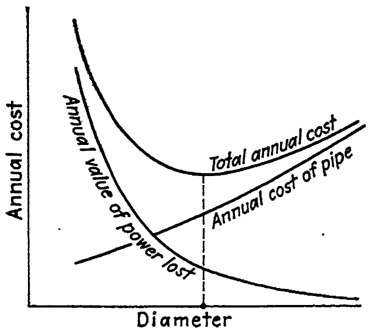


FIG. 155.—To determine economic size.

It must be pointed out that an accurate solution of this problem may be difficult in practice, especially in the case of a water-power plant. The chances are that the rate of flow through the pipe line will not be constant, since the load on the plant will vary, and hence the load factor must be known before one can compute the total amount of power lost per year. Even then it is hard to fix the exact money value of such power lost. Since in a given pipe line the power lost varies as q^3, it is readily seen that the constant flow to be used is the cube root of the mean of the cubes of the actual rates of flow.

It must also be noted that in some cases the economic solution may not be the best. In case the value of a unit of power is small, and the fixed charges are high, the resulting pipe size would be relatively small, and the loss of head large. This means that the velocity of the water would be high, and, as will be seen later, this may cause trouble due to surges and water hammer when any change is made in the flow in governing the turbine. Also, when the loss of head is large the variation in head from full load to no load is large, as may be seen in Fig. 160. This may also be undesirable in the operation of the turbine. Hence for these reasons a larger size pipe may be used.

The higher the pressure in a pipe line the greater the cost of a pipe of given diameter. Thus in Fig. 155 the curve for the

annual cost of pipe is higher with increasing heads, and the most economic diameter is, therefore, smaller. Hence the penstock for a high-head plant should not be uniform but should diminish in size as the head increases, while at the same time the thickness of the metal should increase. For such a case, the values in Fig. 155 should not be for the entire length of the pipe but only for a short length, say 1 ft., and the solution repeated for various intervals. There is a more convenient method of solution of the problem for this case, but the explanation requires too much space to be given here.[1]

For a given pressure the thickness of the walls of a pipe will increase with the diameter, though not always in direct proportion because of manufacturing and other reasons. Also, the unit cost per pound may not be the same for all sizes. But despite these facts it will be found that the weight and the cost of pipe of a given type of construction will vary approximately as the square of the diameter, especially for a moderate range of size. Hence the fixed annual cost of a pipe may be expressed as Ad^2, where A depends upon the length of the pipe; the unit cost; the type of construction; and the percentage assumed for interest, depreciation, etc.

From Eq. (165) it is found that the annual value of the power lost in pipe friction may be represented by B/d^5, where B depends upon the length of the pipe, the annual value of a horsepower, the rate of flow, the density of the fluid, and the value of the pipe friction factor f, which is not, in general, independent of the diameter.

The total annual cost of a pipe line may then be expressed as

$$C = Ad^2 + \frac{B}{d^5},$$

where A and B are not necessarily constants over wide ranges. But for a small range they will be substantially constants, and even for a considerable range of values of diameter they may not vary so much but that average values could be used with small error. Differentiating this equation and equating to zero to determine the mathematical minimum on the assumption that A and B are constants, the result is

[1] It will be found in papers by H. L. Doolittle and the author in *Trans. A.S.M.E.*, vol. 46, p. 1165, 1924.

$$d^7 = \frac{5B}{2A},$$

which gives the most economical diameter for the assumptions made. Even if A and B depart widely from constant values, this equation will yield a value that will be a good approximation and one that may be used as a starting point in the more accurate trial solution.

EXAMPLES

217. The average weight of large riveted-steel pipes in pounds per foot of length is given by $0.045hd^2$, where h is the pressure to which the pipe is subjected in feet of water, and d its diameter in feet. The average cost installed in place is 10 cts. per lb. Assuming the fixed charges to be 10 per cent, what is the value of A for a head of 500 ft. of water? What is the annual fixed cost of a pipe 4 ft. in diameter? *Ans.* 0.225, $3.60.

218. In a certain installation the cube root of the mean of the cubes of the varying rate of discharge during a year is 200 cu. ft. per sec. The annual value of a horsepower is estimated to be $40. The pipe carries water. Assuming $f = 0.014$, compute the value of B. What will be the annual value of the power lost in friction per foot of pipe if the diameter is 4 ft.? *Ans.* 12,800, $12.50.

219. For the case given in Probs. 217 and 218, find the most economical diameter and the corresponding total annual cost per foot of length.
Ans. 5.45 ft., $9.35.

220. A power plant is to be supplied with water at a constant rate of 300 cu. ft. per sec. The penstock is of riveted steel and is 7,000 ft. long. It will be of a uniform diameter. The static head on the plant is 800 ft., and because of this high head the greater portion of the penstock will be made with heavy overlapping plates and with numerous large rivet heads, thus producing a relatively rough pipe. Assuming a constant friction factor $f = 0.020$, fixed charges to be 10 per cent, and the value of a horsepower to be $30 per year, fill in the accompanying table and determine the most economical diameter.

Diameter, in.	Cost of pipe	V	h_f	Annual value of power lost	Annual fixed charges	Total annual cost
70	$230,000	11.21	47.0	$48,000	$23,000	$71,000
80	300,000					
90	360,000					
100	400,000					
110	480,000					
120	560,000					

122. Compound Pipes.—In the case of flow through compound or parallel pipes, such as in Fig. 156, the following fundamental relations exist. The sum of the flow through all the compound pipes equals the total flow in the main. And since the pressures at M and N are common to all the pipes, it follows that the loss of head in each pipe is the same. Or

1. $q_0 = q_1 + q_2 + q_3.$
2. $h = \text{same for all.}$

The loss of head in any section of any pipe is

$$h_f = \left(f\frac{L}{d} + n\right)\frac{V^2}{2g},$$

where n is a factor to account for any minor losses and in long

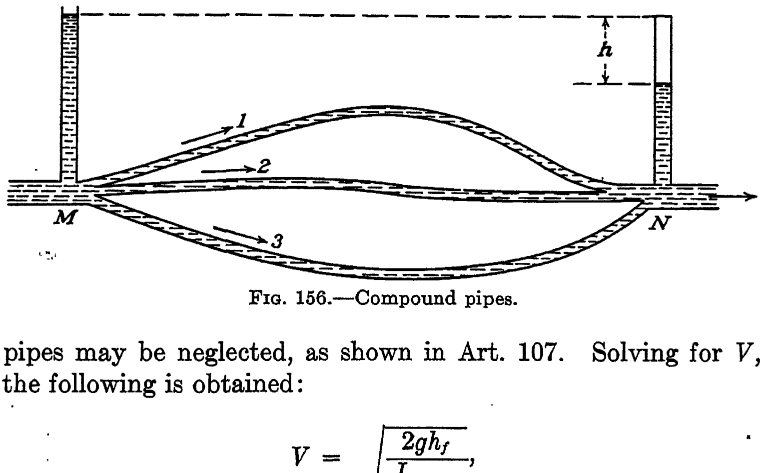


FIG. 156.—Compound pipes.

pipes may be neglected, as shown in Art. 107. Solving for V, the following is obtained:

$$V = \sqrt{\frac{2gh_f}{f\frac{L}{d} + n}},$$

$$q = AV = A\sqrt{\frac{2gd}{fL + nd}}\sqrt{h_f} = K'\sqrt{h_f},$$

as in Eq. (163). In long pipes, also, the difference between h_f and h may be considered to be negligible, so that Eqs. (163) and (164) are practically equivalent. Since K is a constant for any given pipe, its value may be computed in each case, and the following written:

$$
\begin{aligned}
q_1 &= K_1\sqrt{h}, \\
q_2 &= K_2\sqrt{h}, \\
q_3 &= K_3\sqrt{h}, \\
\hline
q_0 &= K_0\sqrt{h},
\end{aligned}
$$

where $K_0 = K_1 + K_2 + K_3$.

If the value of h is given, and all dimensions of the pipes are known, it is then easy to find the rate of discharge in each separate pipe. If the total rate of discharge q_0 be given, the value of h can be computed, and then the flow in each pipe can be found. If the dimensions of one or more of the pipes are unknown, however, a solution by trial may be necessary. If any water is supposed to be withdrawn between M and N, it will then be necessary to combine this problem with that in Art. 123.

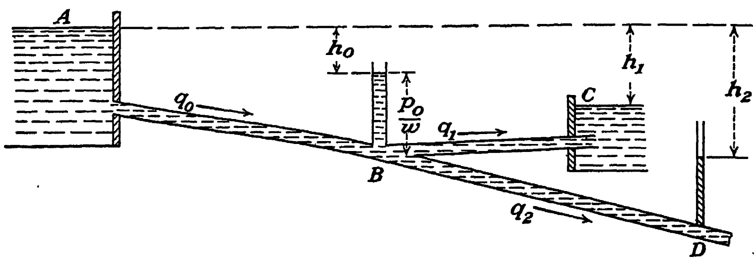


FIG. 157.—Branching pipes.

EXAMPLE

221. In Fig. 156 suppose that water enters at M from a large standpipe and that N is located 50 ft. above a given datum plane. The three pipes are of the following dimensions: 1,200 ft. of 6-in. pipe, 1,000 ft. of 8-in. pipe, and 1,200 ft. of 10-in. pipe, while the diameter at N is 16 in. Assume $f = 0.025$ for each pipe and that velocity is negligible at M but is to be considered at N. If 14 cu. ft. of water per sec. are delivered at N under a pressure head of 100 ft., what must be the elevation of the water surface in the standpipe above the datum plane? *Ans.* 253 ft.

123. Branching Pipes.—Suppose that the fluid flowing in pipe AB in Fig. 157 divides at B, a portion flowing through BC into the reservoir shown, while the rest flows on through pipe BD to a destination not shown. Suppose the pressure at D to be indicated by a piezometer column. (In reality, both branches are similar, since the condition would be practically the same if the

second pipe discharged at D into a reservoir whose surface were the same as the top of the water in this tube.) There are two fundamental relations. The flow in the main AB is equal to the sum of the flow in the branches. Also, the pressure at B is a value common to all three pipes. That is,

1. $q_0 = q_1 + q_2$.
2. p_0/w (or h_0) = common to all.

From Eq. (164), we have

$$q_0 = K_0\sqrt{h_0},$$
$$q_1 = K_1\sqrt{h_1 - h_0},$$
$$q_2 = K_2\sqrt{h_2 - h_0}.$$

These equations may be verified by writing the general equation of energy between A and B, B and C, and C and D. In

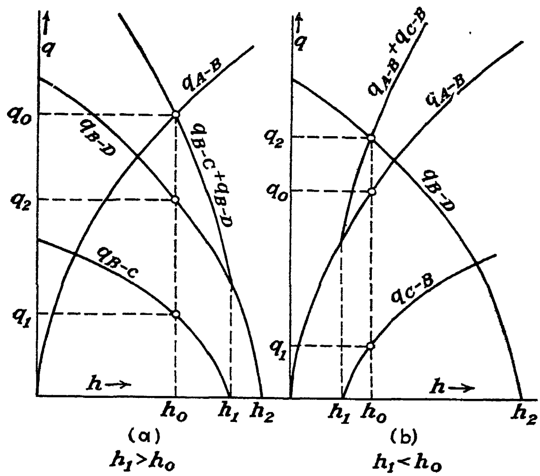


FIG. 158.—Branching pipes diagram.

the case of reasonably long pipes we may disregard the minor losses and also velocity heads at such points as B and D. This problem is not so readily solved as the one in the preceding article because the factor under the radical sign is different for each pipe. Also, in the former case there may be different rates of flow and hence different values of h. But in this case there is only one value of p_0/w or h_0 for equilibrium and hence there can be only one rate of flow, if other dimensions are fixed.

If in Fig. 157 the value of h at B were assumed to vary, while all other dimensions remained constant, the flow in each pipe would vary as a function of h, as shown in Fig. 158. But there is only one specific value of h, namely, h_0, that can satisfy the conditions of the problem. The determination of this value is given by the intersection in Fig. 158 (*a*) of the curves for q_{A-B} and $q_{B-C} + q_{B-D}$, because for this point $q_0 = q_1 + q_2$.

It is also possible for the conditions to be such that h_0 is greater than h_1, for instance, in which case the flow would be from C to B, and then the condition to be satisfied is that $q_2 = q_0 + q_1$. This case is illustrated in Fig. 158 (*b*).

In general, a problem of this type must be solved by trial, but if the value of h_0 is given while some of the other values are left as unknowns, a direct solution may be possible.

EXAMPLE

222. Suppose that in Fig. 157, AB consists of 1,500 ft. of 12-in. pipe, BC of 800 ft. of 6-in. pipe, and BD of 1,200 ft. of 8-in. pipe. Assume $f = 0.025$ for each pipe. The value of h_1 is 20 ft., B is 35 ft. below the level of the water surface at A, and D is 60 ft. below. When the pressure head at B is 25 ft., find the values of the flow in each pipe and the pressure head at D.

Ans. $q_0 = 3.26$, $q_1 = 0.786$, $q_2 = 2.474$, $p_D = 14.12$ ft.

124. Pipe with Laterals.—Assume a pipe main from which fluid is withdrawn by laterals along its course. Then either V or d or both must vary. In such a case the loss of head between any two points may be determined as follows. Differentiating the expression for loss of head, an expression is obtained for the loss of head in any infinitesimal distance dL. Thus

$$dh_f = \frac{f(dL)}{d}\frac{V^2}{2g}.$$

The integration of this between the proper limits of L will give the value of the head lost in that distance. Thus

$$h_f = \frac{1}{2g}\int f\frac{V^2}{d}(dL). \tag{168}$$

If it is possible to express f, V, and d as functions of L, the integration of the foregoing equation will give the value of h_f. If an integration by calculus is not possible, values of fV^2/d may be plotted as a function of L. The area between this curve and the axis for L is the value of the integral.

Special Case.—If the pipe is of uniform diameter, and the laterals are uniformly spaced and may be assumed to take off fluid uniformly along its length, foregoing can readily be integrated. If the velocity of the fluid entering the length considered is V_1, and that leaving it is V_2, while the total length is L, the preceding conditions give

$$\frac{(dL)}{dV} = \frac{L}{V_2 - V_1},$$

since the velocity decreases uniformly along the length of pipe.

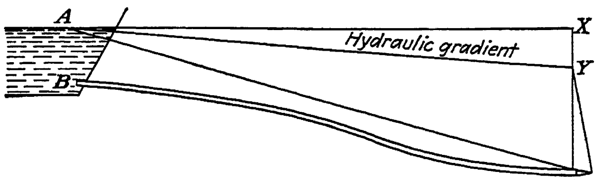


FIG. 159.—Varying hydraulic gradient with different rates of discharge.

Substituting this value of (dL) in Eq. (168),

$$\begin{aligned} h_f &= \frac{f}{2gd} \times \frac{L}{V_2 - V_1}\int_{V_1}^{V_2} V^2 dV \\ &= \frac{1}{3}\frac{fL}{2gd}\frac{V_1^3 - V_2^3}{V_1 - V_2}. \end{aligned}$$

If the terminal of the main is a dead end so that the value of V_2 is zero, this expression is further simplified and indicates that the loss of head is one-third the loss that would exist if the entire amount of fluid entering at (1) flowed clear through the pipe and discharged at (2).

EXAMPLE

223. In Fig. 157 suppose that the branch BD were closed at D and discharged uniformly through laterals along its length. What would then be the pressure head at D, using same data as in Prob. 222? *Ans.* 38.3 ft.

125. Power Delivered by a Pipe.—In Fig. 159 consider a point C which is located near the end of a pipe line. When no flow occurs, owing to the closure of a valve or other device beyond C, the pressure at C is a maximum, being equal to CX. But when flow occurs, the pressure at C drops to the value CY, and the greater the rate of discharge the steeper will be the hydraulic

gradient and the less will be the pressure at C. If the nozzle, or other device beyond C, is removed entirely, making C a point at the very end of the pipe, the pressure will then be reduced to atmospheric or zero gage pressure. In Fig. 160 are shown the decrease in pressure head at C and the increase in velocity head at C as the rate of discharge is caused to increase by opening wider whatever device is below C. Now, the total head at C is the sum of the pressure head and the velocity head, but it is seen to decrease continually with increasing discharge until it reaches a minimum value which is the velocity head when the pipe is wide open.

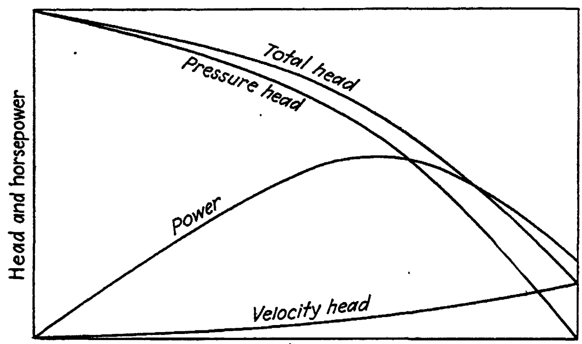


FIG. 160.—Head and power delivered by a pipe.

It has been seen that power is a function of both q and H and may be expressed as wqH_C. In the case under consideration H_C decreases as q increases. When q is zero H_C is a maximum, but the power is zero. And when q is a maximum the power is small because of the small value of H_C. Somewhere between these two extremes the product of these two variables reaches a maximum, as shown by Fig. 160. It can be shown that the power is a maximum when the flow is such that one-third of the static head is consumed in friction, provided h_f varies as V^2 or $h_f = bq^2$. Let z be the static head or total fall.

From Eq. (41),

$$P = wqH_C = wq(z - h_f) = wqz - wbq^3,$$

$$\frac{dP}{dq} = z - 3bq^2 = z - 3h_f.$$

For a maximum value of P,

$$z - 3h_f = 0$$

or

$$h_f = \frac{z}{3}.$$

The efficiency of a pipe line may be defined as the ratio of the power delivered to the power supplied. But power is proportional to head, and hence the efficiency is H_C/z, where $z = CX$ in Fig. 159. In the case of maximum power delivered, one-third of CX has been consumed in friction; hence the efficiency is only $66\frac{2}{3}$ per cent. If economy of water is no object, it would be desirable to transmit power under these conditions, as the cost of the pipe line would be small in proportion to the power delivered. But under usual conditions it is undesirable that one-third of the energy of the water be wasted, and hence such a size of pipe line would be employed that it could deliver the water available with a loss of only a few per cent. With ordinary power-plant practice the efficiency of the pipe lines leading to the turbines is about 95 per cent.

EXAMPLES

224. A pipe line 2,000 ft. long is 5 ft. in diameter. Assume $f = 0\ 02$. If the fall from the reservoir to the end of the pipe is 120 ft., what is the maximum amount of power that the pipe could deliver? *Ans.* 3,200 hp.

225. What amount of power would the pipe in Prob. 224 deliver if its efficiency were 95 per cent? *Ans.* 1,765 hp

226. What size pipe would be required to deliver the water discharged in Prob. 224 if the efficiency of the pipe were to be 90 per cent? *Ans.* 6.4 ft.

126. Pipe Line with Pump.—In case a pump lifts a liquid from one reservoir to another, as in Fig. 161, it not only does work in lifting the water the height z, but it also has to overcome the friction loss in the suction and discharge piping. This friction head is equivalent to some added lift, so that the effect is the same as if the pump lifted the liquid a height $z + h_f$, without loss. Hence the power delivered to the liquid by the pump is

$$W(z + h_f). \tag{169}$$

The power required to run the pump is greater than this, depending upon the efficiency of the pump. Although the pump actu-

ally lifts the liquid a height z, it is said to work against a head h whose value is

$$h = z + h_f. \tag{170}$$

In case the pump discharges a stream through a nozzle, such as in Fig. 162, the liquid has not only been lifted a height z, but it has also received kinetic energy proportional to $V_2^2/2g$, where

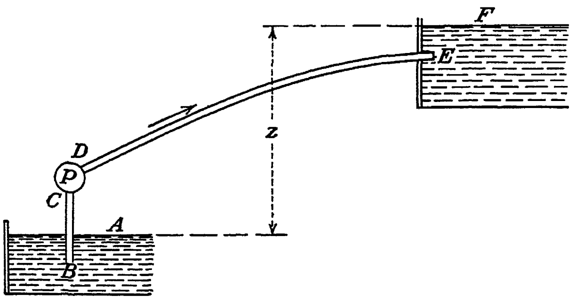


FIG. 161.—Pipe line with pump.

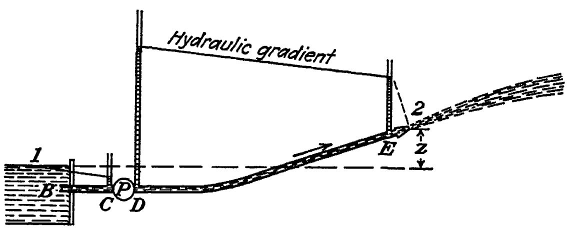


FIG. 162.—Pipe line with pump.

V_2 is the velocity of the jet. Thus the power delivered to the liquid by the pump is

$$W\left(z + \frac{V_2^2}{2g} + h_f\right). \tag{171}$$

And the head against which the pump works is now

$$h = z + \frac{V_2^2}{2g} + h_f. \tag{172}$$

The difference between the two cases in Figs. 161 and 162 is really slight. In Eq. (169) the velocity head at E has been considered to have been lost, while in Eq. (171) the velocity head in the jet has not yet been lost. Thus h_f in Eq. (169) includes the velocity head of discharge, while the h_f in Eq. (171) does not.

EXAMPLES

227. A 10-in. pipe line is 3 miles long. Let $f = 0.022$. If 4 cu. ft. of water per sec. is to be pumped through it, the total actual lift being 20 ft., what will be the horsepower required if the pump efficiency is 70 per cent? *Ans.* 240 hp.

228. In Fig. 161 assume diameter $= 10$ in., $BC = 20$ ft., $DE = 3{,}000$ ft., and $z = 135$ ft. Assume $f = 0.022$. If $q = 7$ cu. ft. per sec., and the pump efficiency is 80 per cent, what is the power required? *Ans.* 340 hp.

229. In Prob. 228, if the elevation of C above the water surface is 13 ft. and that of D is 15 ft., compute the pressures at C and D.

Ans. $p_C = -19.48$ ft., $p_D = +323$ ft.

127. Pipe Line with Turbine.—The type of machine that is usually employed for converting the energy of water into mechanical work is called a turbine. In flowing from the upper body of

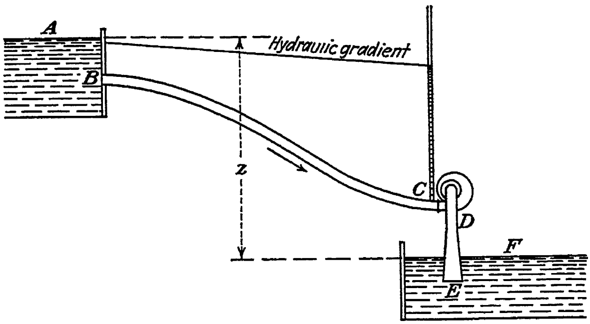


FIG. 163.—Pipe line with turbine.

water in Fig. 163 to the lower, the water loses its potential energy due to the elevation z. This energy, which the water loses, is expended in two ways. A part of it is consumed in hydraulic friction in the pipe, and the rest of it is delivered to the turbine. Of that which is delivered to the turbine, a portion is lost in hydraulic friction within the machine, and the rest is converted into mechanical work.

The power delivered to the turbine is decreased by the friction loss in the pipe line, and its value is given by

$$W(z - h_f). \tag{173}$$

The power delivered by the machine is less than this, depending upon both the hydraulic and the mechanical losses of the turbine. The head under which the turbine operates is

$$h = z - h_f. \tag{174}$$

In the case of a turbine the only loss of head h_f, which is deducted, is that in the supply pipe. The draft tube, as the conduit which leads the water away from the turbine is called, is considered an integral part of the machine, and hence h should cover losses in it as well as in the turbine case itself.

In applying these equations it should be noted that the particular location of the turbine is immaterial so long as it is not set so high above the lower water level that the pressure at the top of the draft tube approaches absolute zero in value. But as long as this is avoided, the turbine can make use of the entire fall to the lower water level by the use of an airtight draft tube. The higher the turbine is situated, within the limit specified, the less the pressure will be at intake, but this is offset by an increased suction on the discharge side.

EXAMPLES

230. In Fig. 163 assume pipe diameter = 12 in., BC = 200 ft., z = 120 ft., and f = 0.0217. The entrance to the pipe at the intake is flush with the wall. (a) If q = 8 cu. ft. per sec., what is the head supplied to the turbine? (b) What is the power delivered by the turbine if its efficiency is 75 per cent?

Ans. (a) h = 112.2 ft. (b) 76.5 hp.

231. A turbine operating under a total fall of 120 ft. (z = 120 ft.) is supplied with water through 300 ft. of 8-in. pipe. Let f = 0.0225. If the rate of discharge is such that 30 ft. of head is lost in friction in the pipe, what will be the power delivered to the turbine?

Ans. 49.2 hp.

128. Equation of Energy with Turbine or Pump.—The general equation of energy, derived in Art. 49, may be applied equally well to a pipe line in which there is any form of turbine or pump between the two sections considered. But the equation should always be applied with the fluid flowing from point (1) to point (2) regardless of the relative positions of these two points. In the preceding article, h_f represents the energy lost by the fluid in pipe friction, while h represents the energy lost by the fluid within the turbine. (Of the latter a part is lost within the turbine in hydraulic friction, and a part is converted into mechanical work, but it is all lost so far as the fluid is concerned.) Hence the following may be written for the turbine:

$$H_1 - H_2 = h_f + h.$$

This is really equivalent to Eq. (174) where A and F correspond to points (1) and (2) in the preceding equation, for $H_A - H_F = z$.

In the case of the pump the h represents energy put into the liquid by the pump between points C and D in Fig. 161 and hence is a negative loss. For the pump, therefore, may be written

$$H_1 - H_2 = h_f - h.$$

This is equivalent to Eq. (170), where $H_A - H_F = -z$; or to Eq. (172), where $H_1 - H_2 = -(z + V_2^2/2g)$.

As an illustration consider a special case where a turbine of known capacity is placed in a pipe line of known dimensions and it is then desired to determine the rate of discharge. Since in the pipe line $h_f = yV^2/2g$, as in Eq. (161), $h_f = Mq^2$ may be written, where M is a constant whose value may be determined from the dimensions of the pipe. It may be shown (Art. 199) that the rate of discharge through any turbine may be expressed as $q = k\sqrt{h}$, where h is the net head utilized by the turbine, and k would be known for a given turbine. Hence may be written $h = (q/k)^2 = Bq^2$, where B is another constant whose value can be determined. Now, referring to Fig. 163,

$$H_1 - H_2 = H_A - H_F = h_f + h,$$
$$z = Mq^2 + Bq^2,$$
$$q = \sqrt{\frac{z}{M + B}}.$$

After q is determined, the net head on the turbine may readily be found, and everything is then known. The method can readily be extended to other combinations.

EXAMPLE

232. Assume the total fall from one body of water to another to be 120 ft. The water is conducted through 200 ft. of 12-in. pipe with the entrance flush with the wall. Let $f = 0.0217$. At the end of the pipe is a turbine and draft tube which discharged 5 cu. ft. of water per sec. when tested under a head of 43.8 ft. in another location. What would be the rate of discharge through the turbine and the net head on it under the present conditions?

Ans. $q = 8$ cu. ft. per sec., $h = 112.2$.

129. Flow of Compressible Fluid with Large Pressure Drop.—The preceding discussion in this chapter applies to a relatively incompressible fluid, that is, to liquids or to gases and vapors where the pressure drop is relatively small compared to the actual

pressure so that the change in density is negligible. In the case of flow of a compressible fluid through a very long pipe line, the variation in pressure may be very great, and this may even be true in a short pipe if the velocity is very high, and the friction loss correspondingly large. In such cases the application of the preceding equations will give results that are only approximations.

In Fig. 164 is shown the variation of pressure for a compressible fluid in a pipe of uniform diameter. Because of the decrease in pressure there is an increase in specific volume with a corresponding increase in velocity. This causes the pressure to drop at a continually increasing rate along the length of the pipe instead of decreasing uniformly, as is the case with an incompressible fluid.

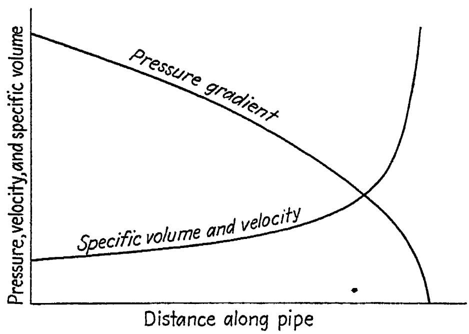


FIG. 164.—Flow of compressible fluid in pipe of uniform diameter.

It may be seen, however, that if the pipe is divided up into short lengths, the pressure gradient can be approximated by a series of straight lines, except for a small portion near the terminal end. For these short lengths, within which the pressure drops are relatively small, the so-called "hydraulic" formulas can be applied, using in each case the average specific volume and average velocity within the section. In case these average values are not known in the beginning, the initial values may be used with but slight error. The solution for a long pipe line is often a "step-by-step" process, and repeated trials are sometimes necessary until all values are consistent.

In general, if the change in density is less than 10 per cent, the previous hydraulic formulas may be used; but if the change is appreciably greater than this, it may be necessary to resort to longer methods of solution. Fortunately, in many practical

cases the pressure drop is not great, but there are also real problems which approach the limiting case shown in Fig. 164.

In only one instance can a correct equation be developed which will give a direct solution and avoid this step-by-step process, and this is for the isothermal flow of a perfect gas. Since many actual cases approximate this rather closely, the result is a very useful one. For a pipe of uniform diameter, it is seen in Fig. 164 that there will be an increase in kinetic energy of the fluid, which will produce a drop in pressure in addition to that caused by friction. Differentiating $V^2/2g$, the result is VdV/g, whose units are in feet of fluid. Multiplying by w, the result $wVdV/g$ is in pounds per square foot. Consequently, the total change in pressure produced by friction in a differential length of pipe dL of diameter d is

$$-dp = wf\frac{(dL)}{d}\frac{V^2}{2g} + w\frac{VdV}{g}. \tag{175}$$

Where w and V are constant, as in the case of an incompressible fluid, this equation at once integrates into its customary form. But for a compressible fluid, w and V are functions of the pressure p. For a perfect gas w may be replaced in terms of p by means of the perfect-gas law $pv = RT$, as it is usually written in thermodynamics, but which will here be written in the form $p = wBT$ to avoid confusion in notation. Here B is used instead of R for the universal gas constant, and w is used instead of specific volume, which is its reciprocal. Hence $w = p/BT$.

From the equation of continuity and by aid of the perfect-gas law, there is obtained $V = W/(wA) = WBT/(pA)$. Substituting for w and V in Eq. (175) and rearranging, we obtain

$$-\left(\frac{2gA^2}{W^2BT}\right)p\,dp = \frac{f}{d}(dL) + \frac{2dV}{V}.$$

In order that this may be integrated, it is necessary that the flow be isothermal, that is, constant temperature, and that the friction factor f be constant. For these assumptions the result for a length L between sections (1) and (2) is

$$p_1^2 - p_2^2 = \frac{W^2BT}{gA^2}\left[f\frac{L}{d} + 2\log_e\frac{V_2}{V_1}\right]. \tag{176}$$

Even this equation may require a trial solution in case V_2 is

unknown. But since the second term is usually small, it may be omitted with but small error. If it is necessary to consider it, a second solution may then be obtained, employing a value of V_2 determined from the results of a preliminary solution. Note that $V_2/V_1 = w_1/w_2$, and sometimes the latter ratio can be determined more readily than the former. Furthermore, as in

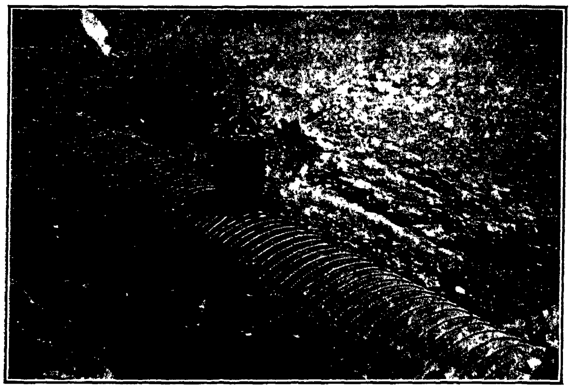


FIG. 165.—Air valve on wooden pipe line. (*Courtesy of Redwood Manufacturers Company.*)

the cases at the beginning of this chapter, a second solution may be necessary if the value of f first employed is not appropriate for the velocity obtained.

EXAMPLES

233. A new steel pipe 10.02 in. in diameter is 10,000 ft. long. It carries superheated steam whose initial values are: pressure, 200 lb. per sq. in. abs.; temperature, 500°F.; specific weight, 0.367 lb. per cu. ft.; velocity, 6,000 ft.

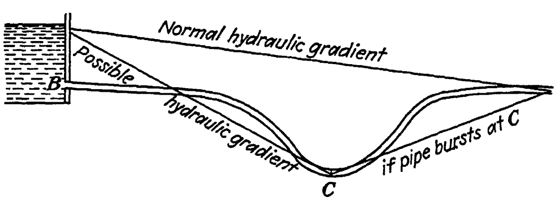


FIG. 166.

per min. Assume $f = 0.0142$ for the conditions given. What will be the pressure at the terminal end of the pipe? (Superheated steam may be treated as a perfect gas with an average value of $B = 83$ for these conditions. Since the drop in temperature was found to be only 17° in this distance, the flow may be assumed to be approximately isothermal.)

Ans. 113 lb. per sq. in. abs.

234. A pipe 12 in. in diameter for which $f = 0.015$ is supplied with 2,000 cu. ft. of air per min. at a pressure of 120 lb. per sq. in. abs. and 60°F. Assuming isothermal flow, what is the drop in pressure in 4,000 ft.?

Ans. 7.5 lb. per sq. in.

FIG. 167.—Cast-iron pipe line.

FIG. 168.—Riveted-steel pipe under head of 1,300 ft. leading to Drum power-house of Pacific Gas and Electric Company in California.

235. What weight of dry saturated steam at 120 lb. per sq. in. will flow through a pipe 8 in. in diameter ($f = 0.015$) if the drop in pressure is 12 lb. per sq. in. per 1,000 ft. of length? (Specific volume = 3.725 cu. ft. per lb.)

Ans. 14 lb. per sec.

236. Compressed air at 200 lb. per sq. in. abs. and 70°F. is supplied to a pipe line 6.25 in. in diameter. At a distance of 1,200 ft. the pressure is

160 lb. per sq. in. abs. Assuming the flow to be isothermal and that the pipe is extremely rough, what is the flow? *Ans.* 11 lb. per sec.

130. Effect of Air at Summit.—In Fig. 63 is shown a pipe line having a "summit" at D, which is above the hydraulic gradient,

FIG. 169.—Old wooden water pipe at New Orleans made from cypress log.

FIG. 170.—Construction of wood-stave pipe. (*Courtesy of Redwood Manufacturers Company.*)

indicating that the pressure at this point is less than atmospheric. In practice this would be avoided, for not only might the excess external pressure cause this portion of the pipe to collapse, but the accumulation of air at this point might interfere with or even stop the flow entirely. All ordinary water carries air in solution

and readily gives it up at a point of low pressure so that air would collect in time, though it were all expelled by some means in the beginning. Therefore in designing a pipe line, whenever any portion of it is found to be above the hydraulic gradient, an attempt would be made to change the profile so that this may be

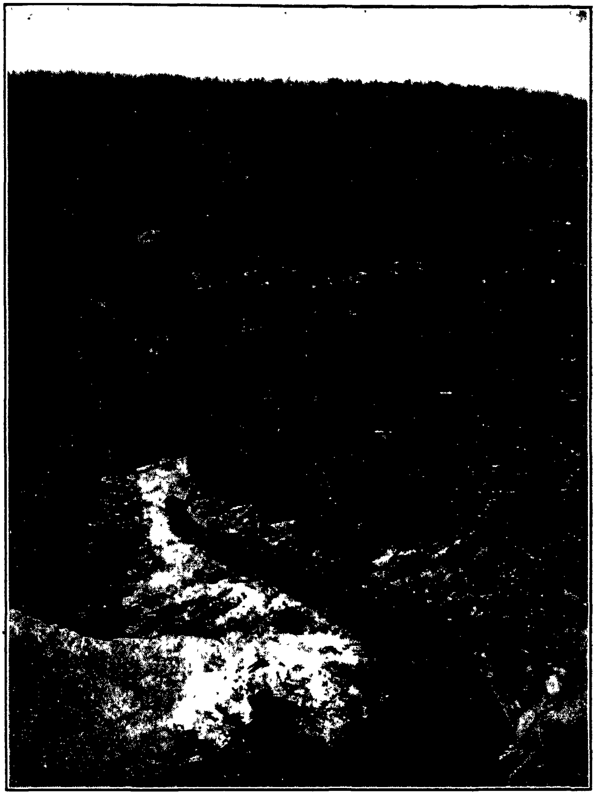


FIG. 171.—Curves in wood-stave pipe. In the Sierra Nevada Mountains of California.

avoided. In case this is impossible, provision must be made for exhausting the air occasionally, if full flow is to be maintained.

If the summit is below the hydraulic gradient, air could still collect, though not so readily, since water under pressure tends to absorb air. But under such conditions it is very easy to release the air, since it will escape if an opportunity is offered it. A valve for such a purpose is shown in Fig. 165. Such valves usually have a float the dropping of which, as air collects

and lowers the water surface, causes a valve to open. When the air escapes, the water level rises, and the float closes the valve again. The valve in Fig. 165 is also constructed so as to admit air into the pipe in case a vacuum should accidentally occur in any way. This will prevent the pipe from collapsing in such an event. In many cases it is highly desirable that pipe lines be furnished with suitable air valves for both these purposes.

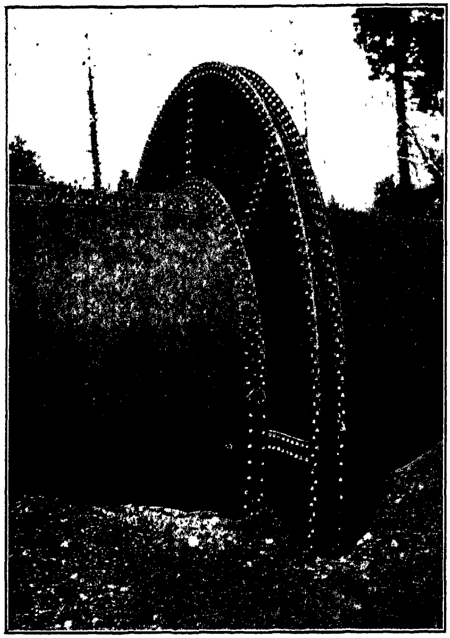


FIG. 172.—Expansion joint in 8.5 ft. riveted-steel pipe under low head.

In Fig. 166 is shown how a vacuum might accidentally occur, when normally the pipe is under a positive pressure. It has been seen that the greater the velocity of flow through a pipe line the less the pressure will be at any point. Hence if some event, such as the bursting of the pipe at C, permits a larger flow of water, the hydraulic gradient will be much steeper than normal. This means that it will be lowered, and it may be lowered sufficiently to be below portions of the pipe, as in Fig. 166.

Also, if the admission of water to a pipe line is shut off by the closure of a gate valve at intake, the water that is already

in the pipe will tend to run out. As no more water can get in to take its place, a vacuum might be created unless air were admitted. Hence, some device is usually provided just below the valve at the intake, and in some cases at other points, to admit air upon such an occasion.

131. Construction of Pipe Lines.—Cast-iron pipes have been employed for the last 200 years and are very satisfactory for ordinary waterworks purposes where moderate heads are employed. They are very durable and require but little attention. While it is sometimes used under higher pressures, cast iron is not considered desirable for heads above 400 ft., nor is it suitable for very large diameters on account of low tensile strength and possible defects in casting. For temporary purposes or for cheaper installations, pipes are sometimes made of very light-weight

FIG. 173.—Expansion joint in high-pressure riveted-steel pipe line.

riveted steel, usually coated with some material in order to enable them to resist corrosion.

For high pressures, cast iron is unsuitable, and steel pipe is used. These may be riveted as in Fig. 168, or they may be welded in special cases. Welded pipe is smoother than riveted pipe. Riveted-steel pipe offers more resistance to flow than a new cast-iron pipe on account of the projecting rivet heads and

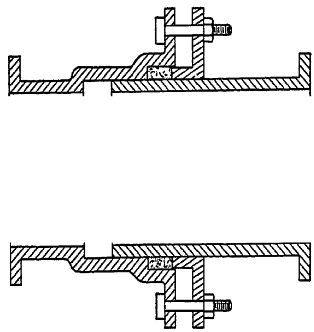


FIG. 174.—Expansion joint.

the overlapping of the plates, but an old riveted-steel pipe and an old cast-iron pipe are about the same, since both become coated alike with tubercles. A steel pipe is not considered so durable as a cast-iron pipe, but for high heads it is necessary to use it.

For heads under 200 or 300 ft., wood-stave pipe offers many advantages. It is cheaper than a metal pipe for the same service.

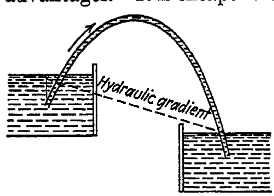


FIG. 175.—Siphon.

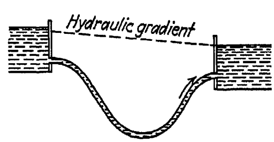


FIG. 176.—Inverted siphon.

The resistance to flow is less than a riveted-steel pipe and about the same as a new, smooth, cast-iron pipe, but it has the advantage that its capacity does not decrease with age. The early types of wooden pipe used were simply hollow logs, as shown in Fig. 169. Some of these were used for many years. Modern wood pipe is generally built up of staves, as shown in Fig. 170. The staves are so arranged that the joints are "broken." In

order to make a watertight joint, a thin steel tongue is inserted in a saw cut across the end of each stave. This piece of steel is slightly wider than the stave so that when the bands are tightened up it will sink into the staves on either side a distance of about 1/8 in. or more. In the life of a wood-stave pipe the encircling metal bands often have to be renewed. It is essential that

FIG. 177.—Riveted-steel siphon. Lake Spaulding development of Pacific Gas and Electric Company.

a wooden pipe be kept filled with water, if it is to have a long life, as wood does not rot rapidly if it is kept continually wet or continually dry. It rots more quickly when it is exposed to alternation in these conditions. The life of a wood pipe is not so long as that of a heavy cast-iron pipe, but it may be as long as that of a steel pipe. However, statistics of these matters are lacking and subject to much dispute. The wood pipe is free from corrosion and from electrolysis and is not attacked by acids in the water. Hence, it is often used for carrying liquids that could not be

handled by a metal pipe. It is possible to introduce broad sweeping curves into a wood-stave pipe without any special devices or fittings, as shown in Fig. 171.

Metal pipes are subject to expansion and contraction due to temperature changes, and provision must often be made for this. In the case of a cast-iron pipe line the amount of play afforded at each joint is usually sufficient. But a riveted-steel pipe line has no such flexibility, and expansion devices may be employed. One type of expansion joint is shown in Fig. 172, which is suitable only for low pressures. It may be seen that the circular plates can spring enough to permit the necessary endwise motion of the pipe. For higher pressures a joint such as in Figs. 173 and 174 may be used.

When water is lifted by a pipe line to a greater height than the initial water level, as in Fig. 175, the pipe is called a siphon. Of course it is necessary to exhaust the air by some means in order to start the flow, and, if the flow is to continue, the air that collects at the summit must be removed from time to time. There are times when such a device cannot be avoided.

By analogy a pipe line such as shown in Fig. 176 is called an *inverted siphon,* and it is usually found where it is necessary to carry water across a valley or depression, as in Fig. 177. It is, however, quite common to call a pipe so situated simply a *siphon.*

132. PROBLEMS

237. A pipe line 850 ft. long discharges freely into the air under a fall of 40 ft. (Assume a projecting pipe at entrance.) (*a*) If $D = 6$ in. and $f = 0.0233$, find the rate of discharge. (*b*) If $D = 12$ in., and $f = 0.0216$, find the rate of discharge.

Ans. (*a*) $q = 1.55$ cu. ft. per sec. (*b*) $q = 8.85$ cu. ft. per sec.

238. Suppose that a pipe line runs from one reservoir to another, both ends of the pipe being under water. Assume that the intake end is non-projecting. If the difference in water levels is 110 ft., the length of pipe 500 ft., and the diameter 10 in., what will be the rate of discharge if $f = 0.022$?

Ans. 11.95 cu. ft. per sec.

239. When the pipe in Prob. 238 is old, assume $f = 0.04$ and that the growth of tubercles has reduced the diameter to 9.0 in. What will then be the rate of discharge? *Ans.* 7.35 cu. ft. per sec.

240. A pump delivers water through 300 ft. of 4-in. fire hose to a nozzle which throws a 1-in. jet. The velocity coefficient of the nozzle is 0.98, and the value of f for the hose may be assumed to be 0.025. The nozzle is 20 ft. higher than the pump. It is required that the velocity of the jet be 70 ft. per sec. What will be the necessary pressure at the pump?

Ans. 45.7 lb. per sq. in.

241. The steel siphon shown in Fig. 177 is 8.5 ft. in diameter. It is 1,900 ft. long and carries 300 cu. ft. of water per sec. What must be the difference in water level at the two ends? (It is arranged as in Fig. 176.)
Ans. 2.8 ft.

242. The pipe line shown in Figs. 173 and 281 has an average diameter of 62 in., is 6,272 ft. long, and the difference in level between the powerhouse and the intake is 1,375 ft. When the pipe delivers 300 cu. ft. of water per sec., what is its efficiency? What is the horsepower delivered to the plant? Assume $f = 0.022$. *Ans.* 93.8 per cent, 44,000 hp.

243. An oil with a kinematic viscosity of 0.005 sq. ft. per sec. and a specific weight of 55 lb. per cu. ft. flows with a velocity of 10 ft. per sec. through a pipe 3 in. in diameter. Find the drop in pressure in lb. per sq. in. per 100 ft. of horizontal pipe. *Ans.* 30.4 lb. per sq. in.

244. California crude oil at 80°F., a temperature at which its kinematic viscosity is 0.0112 sq. ft. per sec., is pumped through a standard-weight seamless-steel pipe ($D = 2.067$ in.). Its specific weight is 59.8 lb. per cu. ft. (*a*) At what maximum velocity would the flow still be laminar? (*b*) What would be the drop in pressure in pounds per square inch per 1,000 ft. of horizontal pipe? (*c*) What would be the drop in pressure if the velocity were three times the value in (*a*)?

245. Compressed air at 100 lb. per sq. in. abs. and 70°F. flows through a pipe which is 2 in. in diameter, and in a distance of 1,000 ft. the pressure has dropped to 15 lb. per sq. in. abs. Assume that the flow is isothermal and that $f = 0.02$. Find the flow in pounds per minute.

246. A pipe line 20 in. in diameter and 500 ft. long discharges freely into the air under a net head of 30 ft. The intake is projecting. Find the drop in the hydraulic gradient at entrance. Let $f = 0.02$. *Ans.* 7.22 ft.

247. Water flows in a 6-in. vertical pipe with a velocity of 10 ft. per sec. The end of the pipe is 3 ft. below the surface of the water. Considering all losses, find the pressure at a point 10 ft. above the surface of the water, when the flow is downward. Let $f = 0.025$. Find the pressure at the same point if the flow is upward.

248. A horizontal pipe 6 in. in diameter discharges under water at a depth 3 ft. below the surface. Find the pressure at a point in the pipe 13 ft. from the end. If the water flows in the reverse direction, and the intake is projecting, find the pressure at this same point. Let $f = 0.025$.

249. In Fig. 63 suppose that the horizontal distance from the intake of the pipe to D is 300 ft. and to H is 900 ft. The vertical distance from the reservoir level to D is 20 ft. and to H is 100 ft. Suppose that at H the water does not discharge freely into the air but the conditions are such that the pressure at H is 52 ft. Draw hydraulic gradient, neglecting slight drop at entrance; plot profile of pipe line (sketching portions BC and EFG at pleasure); and find pressure head at D. Let $f = 0.03$.

250. A pipe runs from one reservoir to another, both ends of the pipe being under water. The intake end is nonprojecting. The length of pipe is 500 ft., the diameter 10 in., and the difference in water levels 110 ft. Let $f = 0.02$. What will be the pressure at a point 300 ft. from the intake, the elevation being 120 ft. less than the surface of the water in the upper reservoir?

251. A pipe line 800 ft. long discharges freely at a point 150 ft. below the water level at intake. The pipe projects into the reservoir. The first 500 ft. is 12 in. in diameter, and the remaining 300 ft. is 8 in. in diameter. Find the rate of discharge, assuming $f = 0.04$.

252. The junction of the two sizes of pipe in Prob. 251 is 120 ft. below the surface of the water level. Find the pressure just above C and just below C, where C denotes the point of junction. Assume a sudden contraction at this point.

253. A jet of water is discharged through a nozzle at a point 200 ft. below the water level at intake. The jet is 4 in. in diameter, and the velocity coefficient of the nozzle is 0.90. If the pipe line is 12 in. in diameter, 500 ft. long, with a nonprojecting entrance, what is the pressure at the base of the nozzle? Assume $f = 0.015$.

254. It is desired to deliver 3 cu. ft. of water per sec. at a point 10,000 ft. distant with a loss of head of 150 ft. What size of cast-iron pipe would be required?

255. Find an expression showing the relation between the discharge and the pipe diameter for a project where the available head is 50 ft. and the length of the pipe line is 2 miles. The pipe is to be of the same size throughout its entire length. Take f for the pipe equal to 0.02. Plot a curve showing this relation, using diameters from 12 to 36 in.

What is the ratio of the discharges for the 36- and 12-in. pipes in this problem? What are the ratios of the areas? Of their velocities?

256. A riveted-steel pipe line 2,000 ft. long is 5 ft. in diameter. The lower end is 140 ft. below the level of the surface at intake and joins on to a turbine at this lower end. If the efficiency of the pipe line is 95 per cent, find the power delivered to the turbine.

257. If the cost of pipe per pound is constant, and the thickness of the pipe is determined solely by pressure conditions, prove that the total annual cost per year is given by $Ahd^2 + B/d^5$, where A and B are constants.

258. Prove that the most economic size of pipe for the foregoing conditions is given by $d = (2.5B/Ah)^{1/7}$.

259. If the thickness of the pipe is required to be greater than the pressure demands, because of structural and other reasons, and is constant, prove that the total annual cost is $Cd + B/d^5$ and that the most economic size is then given by $d = (5B/C)^{1/6}$.

260. Suppose that for a welded-steel pipe the costs per pound, interest rates, value of power, rate of discharge (200 cu. ft. per sec. in this case), etc., are such as to make $A = 0.000437$ and $B = 1{,}325$; find the diameter of pipe at heads of 200, 1,000, and 2,000 ft. *Ans.* 450, 3.59, and 3.25 ft.

261. Find the thickness of metal required in each of the preceding cases in Prob. 260 if the allowable stress is 10,000 lb. per sq. in.

262. Solve Prob. 222, but assuming that the pressure at the junction point is 5 instead of 25 ft., all other data being unchanged.

263. In Prob. 222 if the pressure at the junction point is not fixed, but at D it is assumed to be zero, find rate of discharge in each branch and the pressure at the junction.

264. A 12-in. pipe 10,000 ft. long for which $f = 0.02$ discharges freely into the air at a point 15 ft. lower than the surface of the water at intake. It is necessary that the flow be doubled by inserting a pump. If the efficiency of the latter is 70 per cent, what will be the power required?

265. In Fig. 161 assume diameter = 3 in., BC = 20 ft., DE = 200 ft., and z = 70 ft. The elevation of C above the water surface is 15 ft. Let $f = 0.04$. (*a*) If the pressure at C is to be -25 ft., what is the rate at which water is pumped? (*b*) If the efficiency of the pump is 60 per cent, what is the power required?

266. When a certain pump is delivering 1.0 cu. ft. of water per sec., the pressure gage at D (Fig. 161) reads 20 lb. per sq. in., while a vacuum gage at C reads 10 in. of mercury. The pressure gage is 2 ft. higher than the vacuum gage. If the diameter of the suction pipe is 4 in., and that of the discharge pipe 3 in., find the power delivered to the water.

267. A pump is required to deliver 8 c.f.s. of water from a reservoir to a nozzle 75 ft. below the reservoir. The pipe line consists of 1,000 ft. of 12-in. pipe ($f = 0.025$). Entrance loss coefficient = 1.0. The jet diameter is 4 in. Coefficient of velocity of the nozzle is 0.95. Find the horsepower required to drive the pump with a pump efficiency of 60 per cent.

268. Of the total length of pipe of the preceding problem, 40 ft. is between the reservoir and the pump. The suction and discharge connections of the pump are 3 and 1 ft., respectively, below the surface of the reservoir. Find the pressure on each side of the pump.

269. Water is to be delivered from a reservoir to a second reservoir 500 ft. distant from, and 50 ft. above, the first. A suitable pump capable of delivering a total head of 75 ft. to the water is to be used. With 36-in. pipe ($f = 0.021$) find the rate of flow.

270. A certain turbine was found to discharge 10 cu. ft. per sec. under a head of 64 ft. It was installed at the end of a 12-in. pipe line 500 ft. long, with a flush entrance. Assume $f = 0.02$. The total fall from headwater to tailwater was 40 ft. What will be the rate of discharge, the net head on the turbine, and the power delivered to it?

CHAPTER X

UNIFORM FLOW IN OPEN CHANNELS

133. Open Channels.—An open channel is one in which the stream is not completely enclosed by solid boundaries and therefore has a free surface subjected only to atmospheric pressure. The flow in such a channel is not dependent upon some external

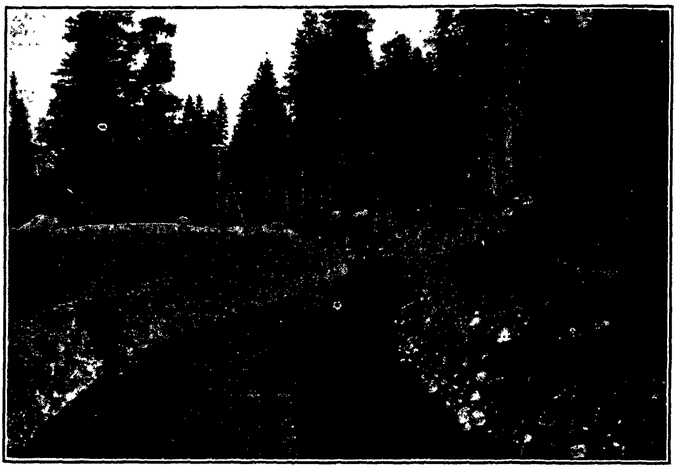


FIG. 178.—Canal of the Pacific Gas and Electric Company with one bank rock lined.

head but rather upon the slope of the channel and of the water surface.

The principal types of open channels are natural streams or rivers; artificial canals; and sewers, tunnels, or pipe lines not completely filled.

The accurate solution of problems of flow in open channels is much more difficult than in the case of pressure pipes. Not only are reliable experimental data more difficult to secure, but there is a wider range of conditions than is met with in the case of pipes. Practically all pipes are round, but the cross sections of open channels may be of any shapes from circular to the irregular forms of natural streams. In pipes the degree of roughness

ordinarily ranges from that of new smooth metal or wood-stave pipes, on the one hand, to that of old corroded-iron or steel pipes, on the other. But with open channels the surfaces vary from that of smooth timber (Fig. 180) to that of the rough and irregular

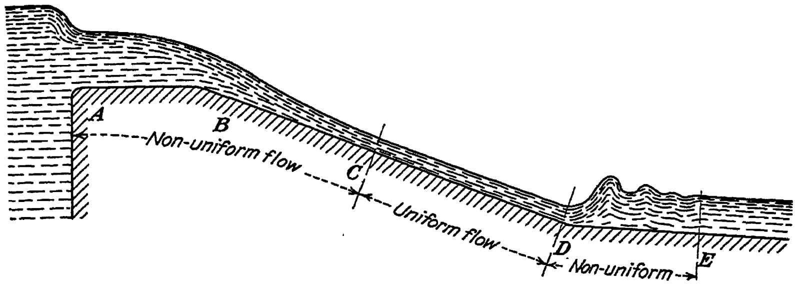


FIG. 179.

beds of some rivers. Hence the choice of friction factors is attended with greater uncertainty in the case of open channels than in the case of pipes.

134. Uniform Flow.—If the shape and size of any water cross section are identical with those of every other section in the length

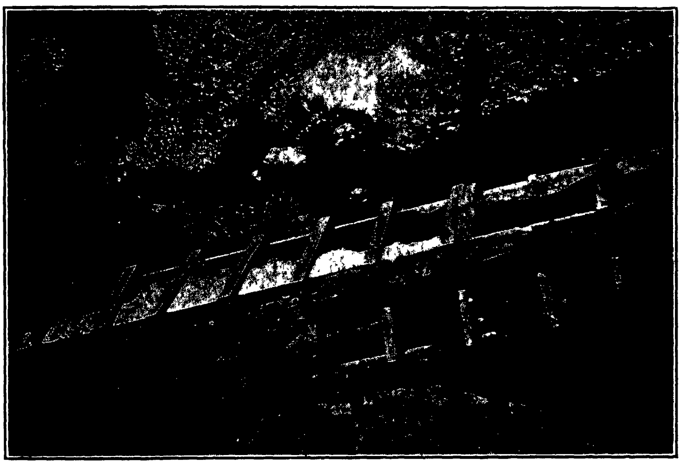


FIG. 180.—Nonuniform flow in wooden flume.

of channel under consideration, the flow is said to be *uniform.* Such cases are shown in Figs. 49 and 178. Uniform flow must not be confused with *steady* flow. The former requires that the conditions at any time be the same from place to place; the latter,

that the conditions at every section be constant with respect to time. We might have steady flow for both uniform and nonuniform flow, as shown in Fig. 179. Uniform flow is obtained only when a channel is uniform for a considerable distance so that the water has a chance to adjust itself. The channel in

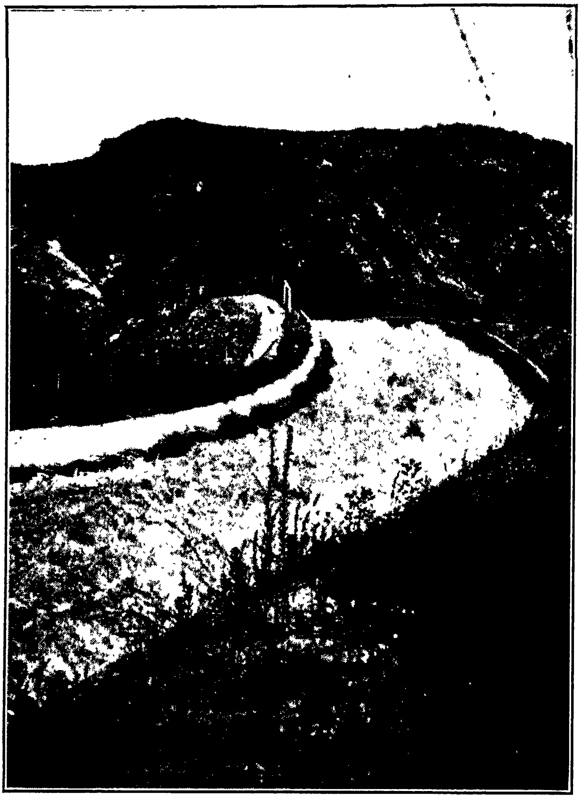


FIG. 181.—Cascade on Los Angeles Aqueduct.

Fig. 180 is uniform, but the flow is nonuniform in the portion shown because the water has just entered it and has not yet attained a condition of equilibrium. The conditions are similar to the lower portion of the channel shown in Fig. 179. On the other hand, the flow is nonuniform in Fig. 181 because the slope of the channel varies.

135. Hydraulic Gradient.—It is quite evident that in the case of an open channel the hydraulic gradient coincides with the

water surface. For if a piezometer tube be attached to the side of the channel, the water will rise in it until its surface is level with that of the water in the channel.

By hydraulic slope is meant the value of S which is given by

$$S = \frac{h_f}{L}. \tag{177}$$

Thus S is the slope of the energy gradient. It applies both to closed pipes and to open channels. For uniform flow in either pressure pipes or open channels the velocity is constant along the length of the conduit, and therefore for this special case the energy gradient and the pressure gradient (or hydraulic gradient) are parallel. Consequently S is then also the slope of the hydraulic gradient, and for an open channel with uniform flow it becomes the slope of the water surface.

In a closed pipe conveying fluid under pressure there is no relation between the slope of the pipe and the slope of the hydraulic gradient. In uniform flow in an open channel the water surface must be parallel to the bed of the stream, and consequently S is then also the slope of the bed.

136. Hydraulic Mean Depth.—By *hydraulic mean depth* or *hydraulic radius* is meant the value of m given by

$$m = \frac{A}{P}, \tag{178}$$

where A is the area of the cross section of the stream, and P is the wetted perimeter; that is, it is the length of the perimeter that is actually in contact with the liquid. In an open channel it does *not* include the distance across the free surface.

In a circular pipe flowing full, $A = \pi r^2$ and $P = 2\pi r$. Hence for this case $m = r/2$ and is obviously not the radius of the pipe. Therefore the term *radius* is misleading and has practically no significance. On the other hand, if a rectangle is constructed with a base P and equal in area to A, then the depth of this rectangle would be m. Also, in the case of a broad but very shallow stream the quantity m is very nearly equal to the mean depth. However, neither of these geometrical interpretations is of any importance, but at least they lend some slight support to the use of the term *mean depth*. What is important is that m is

a ratio which is encountered in the derivation of the fundamental equations for frictional resistance as developed in Art. 93.

It may be seen that for channels of the same area, the wetted perimeter, and therefore m, will vary with the shape of the cross section. Hence m is not only an index of the size of the channel but is also a function of the shape. It is reasonable to suppose that the frictional resistance in channels of the same area would be proportional to the surface or perimeter in contact with the fluid and thus be inversely proportional to m. However, this is strictly true only if the velocity of the fluid adjacent to each portion of the surface is the same in each case. That is, a shape of cross section such that one part of the perimeter is in contact with a fluid whose velocity is high while another part is in contact with a fluid whose velocity is low would not give the same results as another case where the velocity next to all the surface was uniform in value. Hence this ratio m must not be used without due regard to other considerations.

137. Equation for Uniform Flow.—There was derived in Art. 93 an equation for frictional resistance in a conduit of any shape of cross section, which is

$$h_f = \frac{f}{4}\frac{L}{m}\frac{V^2}{2g}.$$

Solving for V^2, the result is

$$V^2 = \frac{8g}{f} m \frac{h_f}{L} = \frac{8g}{f} mS.$$

If a coefficient C is introduced such that $C^2 = 8g/f$ or

$$C = \frac{16.04}{\sqrt{f}}, \tag{179}$$

this reduces to

$$V = C\sqrt{mS}. \tag{180}$$

This is known as Chezy's formula and is widely used both for open channels and for pressure pipes. While either this equation or the one giving a value of h_f can be used for any case, since they are equivalent, and both are applied to closed pipes, custom has

confined the treatment of the open channel to the use of Eq. (180) only.[1]

138. The Coefficient C.—Since C and f are related, as shown by Eq. (179), the same considerations that have been presented regarding the determination of a value for f apply also to the determination of the value of C. However, for open channels the problem is more complex than for circular pipes, as has been mentioned in Art. 133, and hence the selection of a coefficient for an open channel is attended with even more uncertainty than in the selection of one for a closed circular conduit.

It seems obvious, however, that C must be some function of Reynolds number, of the relative roughness, and probably also of shape of cross section. Since the exact effect of the last named is not yet known, the only procedure possible at present is to select f from a consideration of Reynolds number and relative roughness and then determine C from it or, better still, to devise a chart or other means of obtaining C direct.

However, a study of Fig. 141 reveals that as Reynolds number becomes larger its effect upon f diminishes. Also, as the relative roughness increases, the value of f becomes less dependent upon R and more dependent upon the roughness. For a very rough surface the value of f is totally independent of R and depends solely upon the relative roughness.

For a small open channel with very smooth sides the problem of the determination of f or C is the same as that in the case of a pipe. But most open channels are relatively large, compared to most pipes, thus giving larger Reynolds numbers than are commonly encountered in dealing with pipes. Also, in many cases open channels are rougher than pipes. Consequently, for most open channels the influence of Reynolds number upon the coefficient is less than in the case of most pipes. The principal factor, then, in determining the coefficient is the relative roughness, which depends upon the character of the surface and the size of the cross section.

139. Manning's Formula for C.—One of the best as well as one of the most widely used formulas for the determination of the coefficient C is that of Robert Manning, who published it in 1890. Manning's formula is

[1] Instead of Eq. (180) in the form shown, exponential equations are also used, as indicated by Eq. (152) (see Art. 105).

$$C = \frac{1.49}{n} m^{1/6}, \tag{181}$$

where n is to be obtained from Table XI. This formula is seen to comply with the conditions just stated, in that C is a function of the relative roughness, since it involves the size of the cross section in the use of the dimension m and the character of the surface in the use of the factor n.

TABLE XI.—VALUES OF n IN MANNING'S AND KUTTER'S FORMULAS
Prepared by R. E. Horton and Others

Nature of surface	n		$K = 1.49/n$	
	Min.	Max.	Max.	Min.
Neat cement surface	0.010	0.013	149	114
Wood-stave pipe	0.010	0.013	149	114
Plank flumes, planed	0.010	0.014	149	106
Vitrified sewer pipe	0.010	0.017	149	87
Metal flumes, smooth	0.011	0.015	135	99
Concrete, precast	0.011	0.013	135	114
Cement mortar surfaces	0.011	0.015	135	99
Plank flumes, unplaned	0.011	0.015	135	99
Common-clay drainage tile	0.011	0.017	135	87
Concrete, monolithic	0.012	0.016	124	93
Brick with cement mortar	0.012	0.017	124	87
Cast iron	0.013	0.017	114	87
Cement rubble surfaces	0.017	0.030	87	50
Riveted steel	0.017	0.020	87	74
Canals and ditches, smooth earth	0.017	0.025	87	59
Metal flumes, corrugated	0.022	0.030	68	50
Canals, dredged in earth, smooth	0.025	0.033	59	45
in rock cuts, smooth	0.025	0.035	59	43
rough beds and weeds on sides	0.025	0.040	59	37
rock cuts, jagged and irregular	0.035	0.045	43	33
Natural streams, smoothest	0.025	0.033	59	45
roughest	0.045	0.060	43	25
very weedy	0.075	0.150	20	10

For practical use it is better to compute V directly, rather than to determine C separately. Inserting the value of C given by Eq. (181) in Eq. (180), the result is

$$V = \frac{1.49}{n} \sqrt[3]{m^2} \sqrt{S}. \tag{182}$$

The equation is given in this form because of the long association in the minds of engineers of values of n for different types of surfaces. Of course, a single number to replace $1.49/n$ is more convenient, but the values would seem unfamiliar. In Table XI a column is added to give values of $1.49/n$. If this be denoted by K, the formula becomes

$$V = K \sqrt[3]{m^2}\sqrt{S}. \tag{183}$$

Values of $m^{2/3}$ may be found in Table XIV (page 448).

140. Kutter's Formula for C.—The formula for C that has probably been more widely used than any other is that of Kutter and Ganguillet, two Swiss engineers, who proposed it in 1869. This formula is based upon a wealth of data from small artificial canals up to natural streams as large as the Mississippi, and for this reason it is believed to be applicable to a wide range of conditions. But any formula that attempts to cover too large a field must necessarily be a mere average of a number of scattered values, and, though giving approximate values at least for any combination of factors, it cannot be expected to give exact values in individual cases. Hence too great reliance must not be placed upon values given by the use of this or any other such empirical formula.

The Kutter formula involves not only m and n but also the ydraulic slope S. It is

$$C = \frac{41.65 + \dfrac{0.00281}{S} + \dfrac{1.811}{n}}{1 + \left(41.65 + \dfrac{0.00281}{S}\right)\dfrac{n}{\sqrt{m}}}. \tag{184}$$

It is obvious that the formula is very cumbersome, but tables and charts have been constructed to facilitate its use. Inspection of Fig. 182 shows that the formula is very illogical, since it is unreasonable to believe that the coefficient should be independent of S when $m = 3.28$ ft. and vary with S for every other value. Nor is there any reason why it should vary with S in just the opposite way for values of m above and below 3.28 ft. It is seen also that except for very small values of the slope the effect of S upon C is negligible. Furthermore, this term for S was introduced into the formula to make it agree with some early stream gagings on the Mississippi River. It is now known that these are in error,

and consequently the effect of the factor S in the formula, so far as it does affect the coefficient, is to introduce an error. For large values of S the Kutter and Manning formulas will be found to be in good agreement. For small values of S the Manning formula is more accurate. Results similar to Fig. 182 would be obtained with any other value of n besides the one there used.

Kutter's formula has been discussed thus fully because it has been so widely used for so many years and also because the values of n, which are now used in Manning's formula, were originally devised for use in the Kutter formula.

141. Construction of Open Channels.—Inspection of the expression $V = C\sqrt{mS}$ shows that, for a given slope and degree of roughness, the velocity increases as m increases. This is

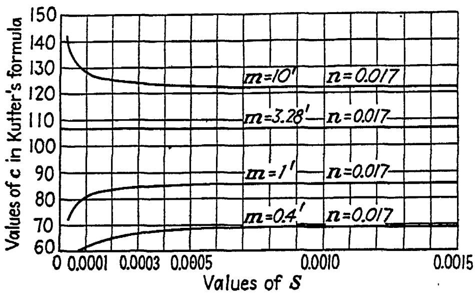


FIG. 182.—Relation between C and S by Kutter's formula.

also accentuated by the fact that the value of C also increases as m increases, or, as shown by Eq. (183), V varies as $m^{2/3}$. Therefore for a given area of water cross section the rate of discharge will be a maximum when m is a maximum. Or for a given rate of discharge the cross-sectional area will be a minimum when the design is such as to make m a maximum.

From Eq. (178) it may be seen that the value of m will be a maximum for a given area when the length of the wetted perimeter is a minimum. Now of all geometric figures, the circle has the least perimeter for a given area. Hence a semicircular open channel will discharge more water than one of any other shape, assuming that the area, slope, and roughness of surface are the same. Semicircular open channels are often built of pressed steel and other forms of metal, but for other types of construction such a shape is impractical.

For wooden flumes the rectangular shape is usually used. Of all rectangles the square has the least perimeter in proportion to area, and hence for an open channel the depth of the water should be half the width.

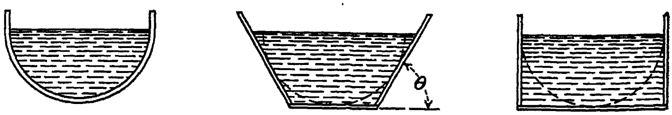


FIG. 183.

Canals excavated in earth must have a trapezoidal section, and of all trapezoids the half hexagon will have the largest value of m. But the angle θ cannot always be made equal to 60 deg. for other

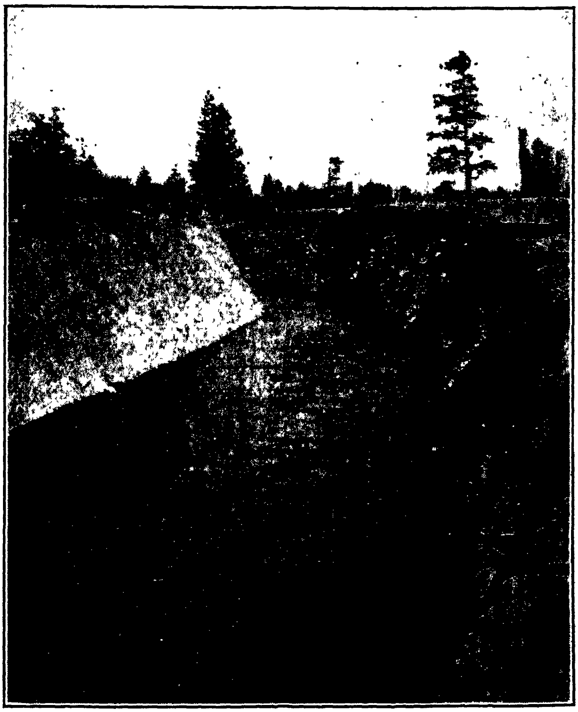


FIG. 184.—Unlined canal with steep banks. In the Sierra Nevada Mountains.

reasons. The slope of the sides must be such that the angle θ is less than the "angle of repose" of the material of which the banks are composed; otherwise the latter will cave in. In Fig. 184

the angle θ is made much greater than 60 deg. in order to save a considerable amount of excavation in a deep cut, the firm character of the soil permitting such steep sides.

Whatever the value of the angle, it will be found that the best proportions will be obtained when the sides are tangent to a semicircle whose center lies in the water surface.

But other forms of cross section are often used either because they have certain advantages in construction or because they are desirable from other standpoints. Thus oval- or egg-shaped sections are common for sewers and similar channels where there may be large fluctuations in the rate of discharge. It is desirable that the velocity, when a small quantity is flowing, be kept high enough to prevent the deposit of sediment, and when the conduit

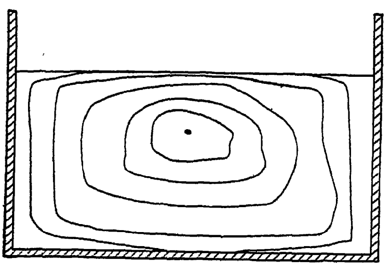


FIG. 185.—Velocity distribution in open channel.

is full the velocity should not be too high on account of wearing the lining of the channel.

142. Stream Gaging.—The determination of the rate of discharge of a stream for any given depth of water is termed *stream gaging*. It may be seen that the rate of discharge of a stream could be computed from the formula $V = C\sqrt{mS}$, if the flow is uniform and the cross-sectional area, the hydraulic mean depth, and the slope of the water surface are known. But a more accurate determination of discharge can be made by measuring the velocity directly, by passing the water over a weir, or by other means.

In a straight portion of an artificial channel the velocity might vary as shown in Fig. 185. These curves are velocity contours or curves of equal velocity. Within the area enclosed by the curve the velocity is higher than that of a point on the curve. Outside the enclosed area the velocity is less than that on the curve. It may be seen that the velocity of the water varies from

side to side and from top to bottom. If there is a bend in the channel, or if the bed is irregular, as in natural streams, these velocity curves are often very irregular and distorted from the forms shown here. It is, therefore, necessary to determine the velocity at a number of different points across the section of the stream.

The instrument that is commonly used for this purpose is the current meter, described in Art. 88. In using the current meter

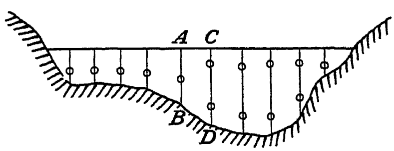


FIG. 186.—Stream gaging.

or any other device it is customary to divide the stream up into sections, as in Fig. 186, and to determine the contour of the bed, so that the area may be computed. If, then, the average velocity is determined for some section such as *ABCD*, the discharge through this section will be the product of this velocity and the area *ABCD*. The sum of all such partial discharges gives the total rate of discharge of the entire stream.

In finding the average velocity in the area *ABCD* it is customary to take it as the average of the velocity measured in the line *AB* and the velocity measured in the line *CD*. But, as shown in Fig. 187, the velocity varies from *A* to *B* or from *C* to *D*, and hence the average velocity in each vertical line should be determined. This might be done by taking a number of observations so that curves similar to that in Fig. 187 could be plotted. But a study of a number of such curves has shown that, in general, the average velocity in a vertical line is found at about 0.6 the depth. Hence if the current meter is set at that depth, the velocity determined by it may be assumed to be the mean velocity. Of course this is only an approximation. To insure a higher degree of accuracy than a single observation could give, measurements are often taken at 0.2 and 0.8 the depth.

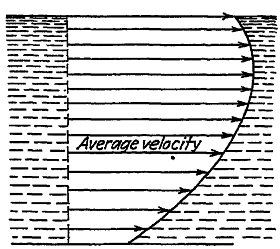


FIG. 187.

The mean of these two values will be approximately the average velocity. Thus, in an actual stream gaging, observations would be made at the points indicated by the circles in Fig. 186. Further details of this topic are not within the scope of this text.[1]

Sometimes floats are used, but such procedure is less accurate. They are, however, often applicable when other methods are not feasible, such as during floods. If surface floats are used, the average velocity may ordinarily be taken as about 0.9 that of the surface velocity. But the velocity at the surface is greatly affected by the wind.

143. Rating Curve.—If a natural stream is to be used for water supply or power purposes, it is necessary to determine the amount

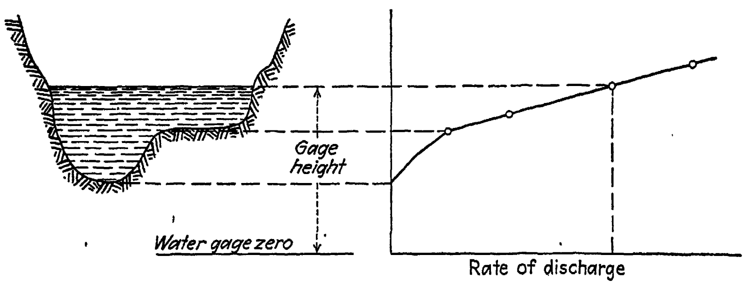


FIG. 188.—Rating curve.

of water that it can be depended upon to furnish. Since the flow will usually be subject to wide fluctuations during a long period of time, it is necessary to make an extended series of observations upon it.

The level of the surface of the water in a stream is called the *gage height* and may be measured above any arbitrary point. Thus the gage height does not necessarily coincide with the depth of the stream.

It is apparent that for a given stream the rate of discharge will be a function of the gage height. If the rate of discharge of the stream is determined for several gage heights, a curve, such as in Fig. 188, may be constructed. This curve is called the *rating curve,* and from it the value of q for any height of water can be obtained.

Thus in making a study of the stream it is necessary to make only a record of the gage heights. From the rating curve the

[1] See HOYT, J. C. and GROVER, N. C., "River Discharge" John Wiley and Sons, Inc., 1912.

quantity of flow can then be determined. This gage height might simply be read and recorded once a day by an observer, or by means of a float and clockwork a continuous record could be obtained which would show all the variations in the flow.

144. PROBLEMS

271. A rectangular flume of timber slopes 1 ft. per 1,000 ft. Compute the rate of discharge if the width is 6 ft. and the depth of water 3 ft. What would be the rate of discharge if the width were 3 ft. and the depth of water 6 ft.? Which of the two forms would require less lumber?

Ans. 114 cu. ft. per sec.

272. A rectangular channel of rubble masonry is 6 ft. wide, the depth of water is 3 ft., and the slope of 1 ft. per 1,000 ft. Compute the rate of discharge and compare with that in Prob. 271. *Ans.* 65 cu. ft. per sec.

273. A semicircular channel of rubble masonry with a slope of 1 ft. per 1,000 ft. will give what discharge when flowing full if its diameter is 6.55 ft.? Compare the cross-sectional areas and amounts of lining required with those in Prob. 272.

Ans. 65 cu. ft. per sec.

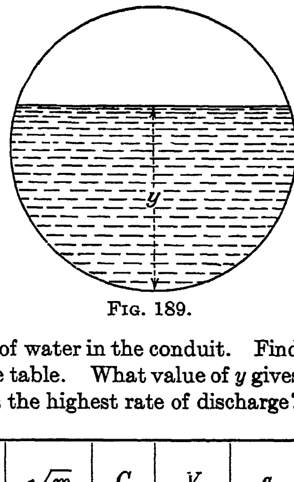


FIG. 189.

274. A circular conduit of concrete ($n = 0.012$) is 10 ft. in diameter and slopes 1.6 ft. per 1,000 ft. (see Fig. 189). The following table gives values of wetted perimeter and area of water cross section for various depths of water in the conduit. Find values of V and q for the various depths in the table. What value of y gives the highest velocity? What value of y gives the highest rate of discharge?

Depth, y	Wetted perimeter	Area, A	m	$\sqrt{m}$	C	V	q
1.0	6.44	4.09	0.635	0.797	115	3.67	15.0
3.0	11.59	19.82					
5.0	15.71	39.27					
8.0	22.14	67.36					
9.0	24.98	74.45					
9.5	26.91	77.07					
10.0	31.42	78.54					

275. In Fig. 190 is shown a cross section of a canal forming a portion of the Colorado River Aqueduct which is to carry 1,600 cu. ft. of water per sec. The canal is lined with concrete for which n is assumed to be 0.014. What must be the grade of the canal and what will then be the drop in elevation per mile? *Ans.* $S = 0.00015$, 0.794 ft.

276. If the flow in the canal in Prob. 275 were to decrease to 800 cu. ft. per sec., all other data, including the slope, being the same, what would be the depth of water?

277. What would be the capacity of the canal shown in Fig. 190 if the grade were to be 1.2 ft. per mile?

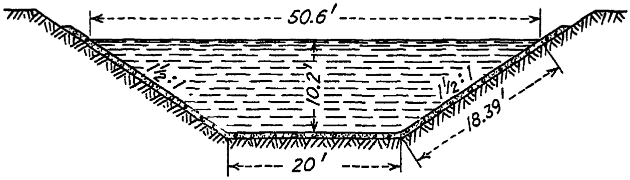


FIG. 190.—Canal section on Colorado River Aqueduct.

278. In Fig. 191 is shown a tunnel section in the Colorado River Aqueduct. The area of the water cross section is 191 sq. ft., and the wetted perimeter is 39.1 ft. The flow is 1,600 cu. ft. per sec. If $n = 0.013$ for a cement lining, find the slope. *Ans.* 0.00065.

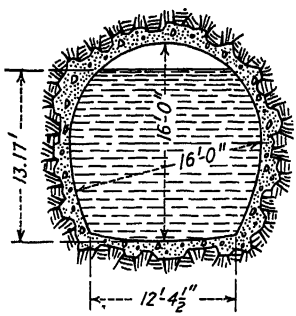


FIG. 191.—Tunnel section on Colorado River Aqueduct.

279. A monolithic concrete inverted siphon on the Colorado River Aqueduct is circular in cross section and is 16 ft. in diameter. Obviously, it is completely filled with water, unlike the case in Fig. 191. If $n = 0.013$, find the slope of the hydraulic gradient for a flow of 1,600 sec. ft. Compare velocities and slopes in Probs. 275, 278, and 279.

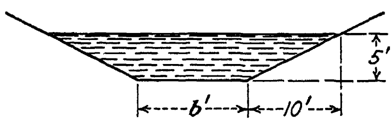


FIG. 192.

280. The amount of water to be carried by a canal excavated in firm gravel is 370 cu. ft. per sec. It has side slopes of 2:1 (horizontal component is two times vertical component), and the depth of water is to be 5 ft. or less (Fig. 192). If the slope is 2.5 ft. per mile, what must be the width at the bottom? (This problem can best be solved by trial.)

281. In Fig. 192, if the rate of discharge is to be 200 cu. ft. per sec., while the velocity is not to exceed 150 ft. per min., what must be the width at the bottom and the drop in elevation per mile? *Ans.* 6 ft., 1.69 ft.

282. A circular conduit of well-laid brickwork when flowing half full is to carry 400 cu. ft. per sec. at a velocity of 10 ft. per sec. What will be the necessary fall per mile? *Ans.* 12.15 ft.

283. A rectangular flume is to be constructed of rough, unsized timber. If given a drop of 10 ft. per mile, what will be the width and depth for most economic solution if it is to discharge 40 cu. ft. per sec.?

Ans. 2 ft. deep and 4 ft. wide.

284. A flow of 84.7 cu. ft. per sec. is to be carried in a rectangular flume made of planed timber, not perfectly true. If the velocity is to be 4 ft. per sec., what will be the proper dimensions and what will be the drop in 3,000 ft.? *Ans.* 3.25 by 6.50 ft., 1.13 ft.

CHAPTER XI

NONUNIFORM FLOW IN OPEN CHANNELS

145. Nonuniform Flow in Open Channels.—As a rule, uniform flow is found only in artificial channels of constant shape and slope, although even under these conditions the flow for some distance may be nonuniform, as shown in Fig. 179. But with a natural stream the slope of the bed and the form and size of the cross section usually vary to such an extent that true uniform flow is rare. Hence the application of the equations given in Chap. X to natural streams can be expected to yield results that are only rough approximations to the truth. In order to apply these equations at all, it is necessary to divide the stream up into lengths within which the conditions are approximately the same. A satisfactory and reliable treatment of the problem of nonuniform flow in natural streams is lacking.

In the case of artificial channels which are free from the erratic irregularities found in natural streams, it is possible to apply analytical methods to the various problems that arise, but in many instances the formulas developed are merely approximations, and in some cases dependence must still be had upon either trial solutions or even purely empirical methods.[1]

In the case of pressure conduits, uniform and nonuniform flow have been dealt with without drawing much distinction between them. This is possible because in a closed pipe the area of the water cross section, and hence the velocity, is fixed at every point. But in an open channel these conditions are not fixed, and the stream adjusts itself to the size of cross section that the slope of the hydraulic gradient requires.

In an open stream the effect of gravity is to tend to produce a flow with a continually increasing velocity along the path, as in the case of a freely falling body. This is opposed by the frictional resistance, and the flow will be either retarded or accelerated

[1] For the treatment of the many types of flow see BAKHMETEFF, B. A., "Hydraulics of Open Channels," McGraw-Hill Book Company, Inc., 1932.

according to whether the friction is greater or less than the gravity component. The frictional force increases with the velocity, while gravity is constant, so it is seen that eventually these two will be in balance provided the channel conditions are constant for a sufficiently great distance. Uniform flow is obtained when these two forces are equal and is thus an equilibrium condition. When the two forces are not in balance, and the flow is either retarded or accelerated along the length of the channel, the flow is nonuniform.

There are two types of nonuniform flow. In one the changing conditions extend over a long distance, and this may be called a gradually varied flow. In the other the change may take place very abruptly, and the transition be confined to a short distance. This may be designated as a local phenomenon. There are numerous forms of each, as will be shown later.

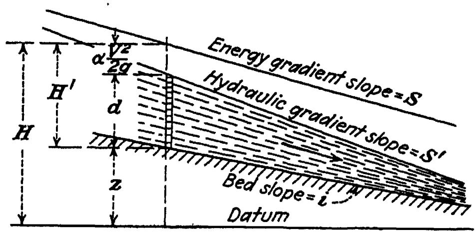


FIG. 193.

146. Energy Equation for Open Streams.—The principal forces involved in flow in an open channel are inertia, gravity, and friction. The first two represent the useful kinetic and potential energies of the liquid, while the third dissipates useful energy into the useless kinetic energy of turbulence and eventually into heat due to the action of viscosity. Referring to Fig. 193, the total energy of the elementary volume of liquid shown is proportional to

$$H = z + d + \alpha\frac{V^2}{2g} \tag{185}$$

where $z + d$ is the potential energy above the arbitrary datum, and $\alpha V^2/2g$ is the kinetic energy, V being the mean velocity in the section. The value of α will generally be found to be somewhat higher in open channels than in closed circular pipes because of the different velocity distribution. It may range from 1.05 to

1.40, and in the case of a channel with an obstruction the value of α just upstream may be as high as 2.00 or even more. Since the value of α is not known unless the velocity distribution is determined, it is often omitted, but an effort should be made to employ it if accuracy is necessary. In the numerical problems in this chapter it will be assumed to be unity.

Differentiating Eq. (185) with respect to x, the distance along the channel, the rate of energy dissipation is found to be

$$\frac{dz}{dx} + \frac{d(d)}{dx} + \frac{d\left(\frac{\alpha V^2}{2g}\right)}{dx} = \frac{dH}{dx} = \frac{dh_f}{dx} = S. \qquad (186)$$

It is seen that $dz/dx = i$, the slope of the bed; while

$$\frac{dz}{dx} + \frac{d(d)}{dx} = S',$$

the slope of the water surface or hydraulic gradient.

The energy equation between two sections (1) and (2) a distance x apart is

$$z_1 + d_1 + \alpha_1\frac{V_1^2}{2g} = z_2 + d_2 + \alpha_2\frac{V_2^2}{2g} + h_f. \qquad (187)$$

Also, since $z_1 - z_2 = iL$ and $h_f = SL$, the energy equation may also be written in the form

$$d_1 + \alpha_1\frac{V_1^2}{2g} = d_2 + \alpha_2\frac{V_2^2}{2g} + (S - i)L. \qquad (188)$$

147. Specific Energy.—The specific energy of an open stream is defined as the energy measured from the bottom of the channel as a base and not from an arbitrary horizontal plane. Thus it omits the z term in Eq. (185) and is

$$H' = d + \alpha\frac{V^2}{2g}. \qquad (189)$$

It is useful because it can be employed to investigate the possible flows at one specific point.

It is evident that for uniform flow the value of H' is constant along the channel, and energy loss is measured solely by the decrease in z, which, in turn, is equal to the drop in the water surface. But for nonuniform flow it will be found that H' may decrease between one section and another because of energy losses which are not manifested by a change in z.

Since $V = q/A$, Eq. (189) can be written as

$$H' = d + \frac{\alpha q^2}{2gA^2}. \tag{190}$$

For any given shape of cross section, A will be a function of the stream depth d; and if for any constant value of q a series of values of d is assumed, corresponding values of H' are determined. If the relation between d and H' is plotted as shown in Fig. 194, it may be seen that, in general, for any one value of H' there are two possible stages of flow, that is, two possible values of d. Thus

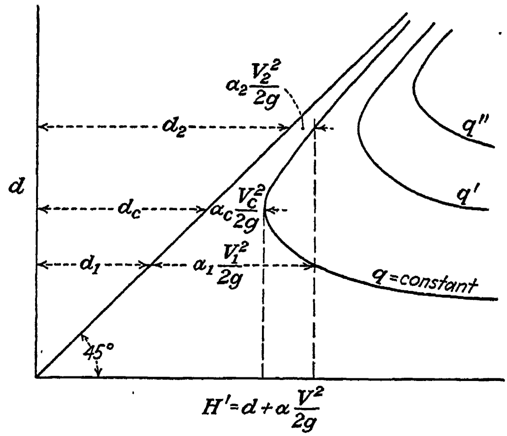


FIG. 194.—Specific-energy diagram.

for the same state of energy the stream may have a low depth and a high velocity or a large depth and a small velocity.

EXAMPLES

285. Using the dimensions for Fig. 189 as given in Prob. 274, compute the specific energy for a constant flow of 120 cu. ft. per sec. at depths of 1, 3, 5, 8 ft., assuming $\alpha = 1$. *Ans.* 14.4, 3.568, 5.145, 8.049.

286. Do the same as in Prob. 285 but for a constant flow of 100 cu. ft. per sec.

148. Critical Velocity and Critical Depth.—It is seen in Fig. 194 that there is one value of the depth for which the specific energy is a minimum. This is determined by

$$\frac{dH'}{d(d)} = 0 = 1 + \frac{d}{d(d)}\left(\frac{\alpha q^2}{2gA^2}\right). \tag{191}$$

In order that this differentiation may be completed, it is necessary to apply it to some specific shape. Assuming a rectangular cross section of uniform depth d and width b, we have

$$\frac{d}{d(d)}\left(\frac{\alpha q^2}{2gb^2}\frac{1}{d^2}\right) = -1,$$

which gives

$$\alpha q^2 = gb^2 d_c^3,$$

from which

$$\frac{\alpha_c V_c^2}{g} = d_c$$

or

$$V_c = \sqrt{\frac{gd_c}{\alpha_c}}. \qquad (192)$$

This shows that in a rectangular channel the water flows with a minimum specific energy if the velocity head is one-half the depth.

The result in Eq. (192) applies only to a rectangular channel, but a corresponding value could be determined for any other shape. It is merely necessary to replace d_c with $d_c' = A/b$, where b is the surface width. The value of d_c is also found by plotting the specific-energy diagram such as in Fig. 194. It will generally be found that the velocity is fairly uniform under this condition, so that the value of α_c is very nearly unity.

This critical velocity is not the same thing as the other critical velocity which divides laminar from turbulent flow, but unfortunately custom has given us a duplication of names. It does, however, divide one type of flow from another, as will shortly be seen, and is thus of great significance.

It may be shown, though it will not be proved here, that the velocity of propagation of a disturbance or wave in still water with a free surface is given by

$$C = \sqrt{gd}\left(1 + \frac{3}{4}\frac{h}{d}\right) \qquad (193)$$

where d is the actual depth, whatever it may be, and h is the height of the wave. Since h/d is usually small, it may be neg-

lected, so that approximately

$$C = \sqrt{gd}. \tag{194}$$

Thus if a stone is dropped into still water, ripples will move radially in all directions with a velocity as given by this expression. (This is not the velocity of an ocean wave.)

From Eq. (194) it is seen that the velocity of a wave will increase as the depth of the water increases. But this is the velocity in still water. If the water is flowing, the actual speed of travel will be the resultant of the two velocities. When the stream is flowing at its critical depth d_c, the stream velocity and the wave velocity will be equal. This means that when the surface is disturbed from any cause the wave so produced cannot travel upstream. If the disturbance is a permanent one, such as that produced by an obstruction to the flow or a change in the channel, the wave remains stationary and is therefore called a standing wave.

When the depth of the stream is greater than the critical depth, and the velocity of flow is therefore less than the critical velocity, the wave velocity C is then greater than the critical value owing to the increase in d. Consequently, any surface disturbance will be able to travel upstream against the flow. Hence the entire flow picture is dependent upon whether the stream velocity is smaller or larger than the critical velocity. The situation is analogous to that of the acoustic velocity as described in Art. 90. The standing wave which exists because of a permanent disturbance, when the flow velocity is above the critical, will be at such a direction that $\sin \beta = C/V = \sqrt{gd}/V$ where β is the angle between the direction of flow and the wave front.[1]

EXAMPLES

287. Prove that for a channel of rectangular cross section the critical depth is two-thirds of the value of the specific energy, if α is assumed to be unity.

288. Prove that for a channel of rectangular cross section the critical depth $d_c = \sqrt[3]{q^2/gb^2}$.

289. A rectangular channel 10 ft. wide carries a flow of 200 cu. ft. per sec. Find the critical depth and the critical velocity.

Ans. 2.33 ft., 8.67 ft. per sec.

[1] Ippen, A. T., and R. T. Knapp, "A Study of High-velocity Flow in Curved Channels of Rectangular Cross Section." *Trans. Amer. Geophys. Union*, 17th Ann. Meeting, part 2, July, 1936.

290. A trapezoidal canal with side slopes of 2:1 has a bottom width of 10 ft. and carries a flow of 600 cu. ft. per sec. Find the critical depth and the critical velocity. *Ans.* 3.75 ft., 9.15 ft. per sec.

149. Hydraulic Jump.—When flow at a velocity above the critical has its velocity reduced to a value less than the critical because of an obstruction in the stream or a change in slope or other channel condition, there is produced in the surface curve a discontinuity known as the *hydraulic jump*. This is shown in Fig. 195, where it is seen that there is an abrupt increase in the height of the water surface.

That this must result in a discontinuity may be seen from Fig. 194, where even without friction loss it is impossible to pass from stage (1) to stage (2) by passing through the critical depth, because this would necessitate a gain in specific energy in attain-

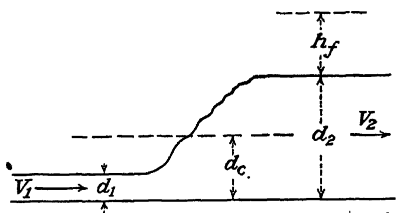


FIG. 195.—Hydraulic jump.

ing stage (2). In reality, there must be a loss of energy because of the turbulence produced, so that the final stage must be lower than shown in Fig. 194.

The equation for the height of the hydraulic jump will be derived for the case of a horizontal channel bottom. It is assumed that the friction forces acting are negligible because of the short length of channel involved and because the shock losses are large in comparison. The key to the solution is the law of conservation of momentum, which is that the time rate of change of momentum of a flowing stream must be equal to the components of all forces acting. Since the channel is horizontal, the gravity forces will have no component in the direction of flow. Hence we have

$$\frac{w}{g}q(V_2 - V_1) = w\int^{A_1} y_1\,dA - w\int^{A_2} y_2\,dA,$$

where y_1 and y_2 are variable distances from the surface to elements

of area in sections (1) and (2) of Fig. 195, respectively. It is seen that each integral is equal to $\bar{y}A$, the statical moment of the area about an axis lying in the surface, where $\bar{y}$ is the vertical distance from the surface to the center of gravity of the area. Hence, after rearranging, we have

$$\frac{w}{g}qV_2 + w\bar{y}_2A_2 = \frac{w}{g}qV_1 + w\bar{y}_1A_1. \tag{195}$$

That is, the momentum plus the pressure on the cross-sectional

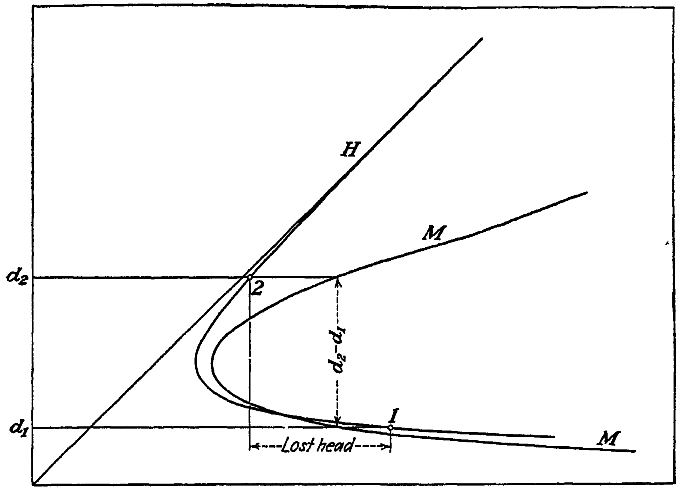


FIG. 196.—Solution for hydraulic jump.

area is constant, or, dividing by w,

$$M = \frac{q^2}{Ag} + A\bar{y} = \text{constant}. \tag{196}$$

A curve of values of M for different values of d is also plotted on the specific-energy diagram shown in Fig. 196. Since loss of energy does not affect the momentum, the latter is the same after the jump as before and therefore any vertical line serves to locate two conjugate depths d_1 and d_2.

Thus in Fig. 196 the line for the initial water level d_1 intersects the M curve as shown and thus gives the value for M which must be the same after the jump. This then serves to fix the value of d_2 and also determines the energy loss involved, all of which is shown in the diagram.

For the rectangular channel all of this can be handled also by straight analytical methods without much labor.

If the channel bottom slopes, the problem is much more complicated, and no simple method of solution has yet been devised. However, there are empirical procedures and also methods of trial solution which have been devised.

There are further problems of determining the location in the channel at which the jump will occur and also of obtaining the distance involved in the transition, but it is beyond the scope of the present treatise to go into these matters.[1]

EXAMPLES

291. Prove that for a channel of rectangular cross section the depth after the hydraulic jump is $d_2 = -\frac{d_1}{2} + \sqrt{\frac{2V_1^2d_1}{g} + \frac{d_1^2}{4}}$.

292. In a rectangular channel 10 ft. wide with a flow of 200 cu. ft. per sec., the depth is 1 ft. If a hydraulic jump is produced, what will be the depth after it? What will be the loss of energy? *Ans.* 4.51 ft., 2.41 ft.

293. If there is a flow in the canal in Prob. 292 with a depth of 2 ft., what will be the height after a hydraulic jump? What will be the loss of energy? *Ans.* 6.1 ft., 2.72 ft.

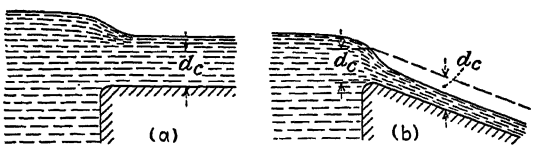


FIG. 197.—Hydraulic drop.

150. The Hydraulic Drop.—When water enters a canal in which the depth is greater than the critical value, there will be a drop in the water surface due to the velocity head and also to any friction loss at entrance. Such a case is shown in Fig. 197 (a). But if the depth is less than the critical value, the water level at entrance will drop to the critical depth and no farther, no matter how low the water level may be in the stream below. The maximum flow in the canal is then limited by this factor. The surface curve in Fig. 197 (b) will have a point of inflection

[1] ELLMS, R. W., "Computation of the Tail-water Depth of the Hydraulic Jump in Sloping Flumes," *Trans. A.S.M.E.*, HYD-50-5, 1927–1928; "Hydraulic Jump in Sloping and Horizontal Flumes," *Trans. A.S.M.E.*, HYD-54-6, November, 1932.

HINDS, JULIAN, "The Hydraulic Jump and Critical Depth in the Design of Hydraulic Structures," *Eng. News-Record*, Nov. 25, 1920.

at about the entrance section; and although the curve downstream may be changed with different channel conditions, the portion of the curve upstream from this point of inflection remains unaltered.

If the surface in Fig. 198 is horizontal, and if the flow along it is in parallel lines so that centrifugal effect does not alter the normal hydrostatic pressure variation, then the depth of flow along it may be made to be the critical depth by a suitable arrangement of the channel. Critical flow is produced in such a case by a reduction in width of cross section or by a change in the bottom. Since the critical depth is a known function of the flow, such an arrangement may be used as a meter.

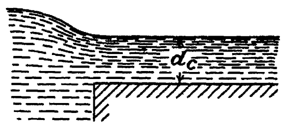


FIG. 198.—Critical depth.

151. Drop-down Curve.—A modification of the critical depth principle is in the drop-down curve shown in Fig. 199. This

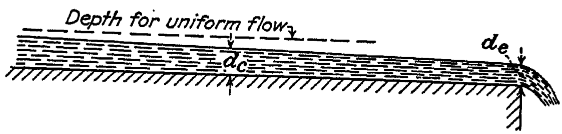


FIG. 199.—Drop-down curve.

differs from all the preceding cases in that it is an example of a gradually varied flow. For if the channel bottom has a small slope, and if the depth is above the critical, then, with a free overfall at the discharge end, the surface curve will gradually rise upstream and approach the depth for uniform flow at an infinite distance. Practically, because of slight wave action and other irregularities, the distinction between the drop-down curve and the curve for uniform flow disappears within a finite distance.

Since friction produces a constant diminution in energy in the direction of flow, it is obvious that at the point of outfall the energy must be less than at any point upstream. Since the critical depth is the value for which the energy is a minimum, then this should apparently be the depth of the stream at this point. However, the value for critical depth is derived on the assumptions that the water is flowing in straight lines, so there is no centrifugal effect to alter the pressure over the section, and also that the pressure on the stream bed is equal to the hydrostatic

depth. Such are the conditions in many cases. But in the case of the free fall, shown in Fig. 199, the curvature of the streamlines reduces the pressure below that corresponding to the depth. And also, if the underside of the stream is open to the atmospheric pressure, then the pressure on the channel bottom at that point will be atmospheric pressure and independent of the depth of the water.

When these changed conditions are considered it is found that the specific energy can be even less than the minimum value found for the preceding case, and thus the depth at the end is not d_c but a smaller value d_e. By making certain assumptions as to the pressure over the section, it has been shown that

$$d_e = 0.667 d_c \tag{197}$$

in the case of a horizontal stream bed. Certain experiments have given results that agree very closely with this value and furthermore show that the normal critical depth d_c is not attained until the distance upstream is about $18d_e$.[1]

If, instead of a free overfall, the conduit discharges into water whose surface is higher than the level indicated by Fig. 199, then of course the surface at discharge must rise to that value, but it can never be lower than d_e.

It is impossible to have uniform flow in a channel whose bed is horizontal, and it is in this case that the drop-down curve is the most noticeable. With increasing values of the slope, higher values for uniform velocity are found, and consequently lower values of the depth of water. When the slope is such that the depth for uniform flow is equal to or less than d_c, uniform flow exists up to a point very near the outlet. Thus the drop-down curve is found only where the depth of the stream is greater than the critical depth.

EXAMPLES

294. A portion of the Los Angeles outfall sewer is approximately a circular conduit 5 ft. in diameter and with a slope of 1 ft. in 1,100 ft. It is of brick, for which $n = 0.013$. What would be its maximum capacity for uniform flow? If it discharges 120 cu. ft. per sec. with a depth at the discharge end of 3.15 ft., how far back from the end must it become a pressure conduit unless the size or the slope is changed?

The solution of Example 274 shows that the discharge is a maximum when the depth is 0.95 times the diameter, or 4.75 ft. in this case. The value of

[1] O'BRIEN, M. P., *Eng. News-Record*, Sept. 15, 1932.

A is then 19.30 sq. ft., and m is 1.435 ft. For uniform flow the velocity may be found to be 4.37 ft. per sec., and the discharge 84.3 cu. ft. per sec.

For a discharge of 120 cu. ft. per sec. the depth at the mouth of the sewer is 3.15 ft. Proceeding upstream, the depth increases until it becomes 4.75 ft. The problem is to find the distance to this section. This may be solved by a step-by-step method with Eq. (188). The procedure is to assume a depth of water and then compute the distance L to that section from some other section previously determined. Such a solution is greatly facilitated by a tabulation, such as here presented. Fill in the table and complete the solution. *Ans.* 718.5 ft.

d	A	m	C	V	$V^2/2g$	$S = \frac{V^2}{C^2m}$	S average	$\Delta\left(d + \frac{V^2}{2g}\right)$	$S - i$	L	ΣL
3.15	13.1	1.42	121	9.20	1.31	0.00406	....	000	....	0	0
3.50	14.7	1.48	122	8.17	1.04	0.00302	0.00354	0.080	0.00263	30.4	30.4
3.75	15.8	1.51	122	7.59	0.895	0.00257	0.00280	0.105	0.00189	55.5	85.9
4.00	16.9	1.53	123	7.13	0.790	0.00221	0.00239	0.145	0.00148	98.0	183.9
4.25	17.8	1.52	122	6.75	0.709						
4.50	18.6	1.49	122	6.45	0.645						
4.75	19.3	1.44	121	6.22	0.600						

295. In Example 294 assume the rate of discharge to be 84.3 cu. ft. per sec. and find the depth of water at the section of free outfall, as in Fig. 199, and the distance from the mouth at which the depth would equal 4.50 ft.

152. The Backwater Curve.—The specific-energy diagram (Fig. 194) shows three cases of nonuniform flow. One is the hydraulic jump, which is produced when the flow changes so that the stream depth passes through the critical value. That is, the flow condition jumps from the lower branch of the specific-energy diagram to the upper. Such a case has been seen to result when a flow at less than the critical depth is slowed down by some means to a velocity less than the critical. Among the ways in which such an effect may be produced is the obstruction of the high-velocity stream by a dam or some equivalent structure.

If, however, the velocity is not reduced to the critical value, so that the surface is not raised to the critical height, then the condition is shown by proceeding along the lower branch of the specific-energy diagram. That is, as the depth increases, the velocity decreases as does the specific energy. The result will be a smooth surface curve extending over a long distance by contrast with the hydraulic jump which is purely local.

The more common case is where the depth is already above the critical value and is increased still further by a dam. Such a case is shown in Fig. 200 but drawn for a natural stream with an irregular stream bed. This case is found on the upper branch of the specific-energy diagram, for here, also, as the depth increases the velocity diminishes without any abrupt transitions, so that again a smooth surface curve is obtained.

All three of the cases just described are technically backwater curves, but in practice the term is usually understood to apply principally to the last named. If this case were flow in an artificial channel with a constant slope of the bed, the backwater curve would be asymptotic at infinity to the surface for uniform flow. But the problems that are usually of more important interest are those concerned with the effect of a dam on a natural stream. This also raises the surface for a very great distance

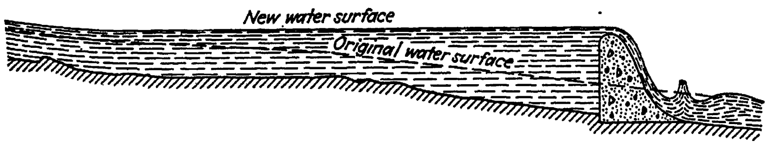


FIG. 200.—Backwater curve.

upstream, and it is important to know how much the water surface will be raised at certain points by the erection of a dam.

For an artificial channel where conditions are uniform, save for the variation in the depth of the water, this problem may be solved by the use of Eq. (188) in a manner identical with that shown in example 294. For a natural stream, such as that in Fig. 200, with varying slopes of bed and different cross sections along its length, the solution is not so direct, because the form and dimensions of a cross section of the stream cannot be assumed, and then the distance to its location computed. Since at different distances upstream there exist various slopes and types of cross sections, the value of L in Eq. (188) must be assumed, and then the depth of stream at this section must be computed by trial.

In view of the fact that for an irregular natural stream so many approximations must be made as to the actual conditions, the refinements of Eq. (188) are hardly justified, and it is fully as satisfactory to apply the simple equation for uniform flow

$V = C\sqrt{mS}$. In order to do this, the stream must be divided up into various lengths within which the flow may be assumed to be fairly uniform. Then for each length average values of m and of V are used, and the value of S is determined by trial.

It is also possible to solve such a problem by certain more analytical methods and avoid this rather tedious trial solution, but space does not permit the explanation of these other methods.

EXAMPLES

296. When the flow in a certain natural stream is 7,600 cu. ft. per sec., it is required to find the elevation of the water surface at different sections upstream from a certain initial point. Computations could be made downstream just as well, but it is customary to start at the dam, since the conditions there are usually assumed to be known. A survey of the channel shows that conditions are fairly similar for a length of 1,500 ft. upstream from the initial point, and then beyond that is another stretch of 2,200 ft., and so on. Assuming a rise in the water surface in the distance of 1,500 ft. to be 0.20 ft., a study of the stream bed shows the average value of the area to be 3,200 sq. ft., and the average perimeter to be 349 ft. This gives an average velocity of 2.45 ft. per sec. and an average m of 8.87 ft. The computed loss of head is then $SL = LV^2/C^2m = 0.283$ ft., which is greater than that assumed. Hence assume a larger value and repeat. Complete the following table and find the rise in elevation in the first 1,500 ft.

Ans. 0.265 ft.

Assumed rise SL	A average	Perimeter average	m average	V average	Computed $SL = LV^2/C^2m$
0.20	3,100	350	8.86	2.45	0.283
0.25	3,180	359	8.86	2.39	0.269
0.26	3,190	360	8.86		
0.265	3,200	361	8.86		
0.27	3,220	363	8.86		
0.28	3,230	364	8.86		

In a similar manner the rises in other lengths may be computed, and the sum of all of them up to the desired point will give the elevation at that point above the initial.

297. The slope of a stream of rectangular cross section is $i = 0.0002$, the value of C is 78.3, $m = 0.8y$, and the flow per foot of width of the stream is 88.55 cu. ft. per sec. Find the depth for uniform flow. If a dam raises the water level so that at a certain section upstream the increase is 5 ft., how far from this latter section will the increase be only 1 ft.? How far downstream will the increase be 10 ft.? *Ans.* 20 ft., 66,200 ft., 41,000 ft.

CHAPTER XII

UNSTEADY FLOW

153. Unsteady Flow.—The other chapters in this text are practically confined to the problems of steady flow, but in the present chapter there will be a brief analysis of some of the fundamental types of unsteady flow. In uniform flow we deal with an equilibrium condition where all quantities are constant with respect to space. In steady flow we have an equilibrium condition where all quantities have become constant with respect to time. As has been stated previously, turbulent flow is unsteady flow in the strictest sense, but we shall not consider it in that way here. If the temporal mean values are constant over a period of time, it may be called mean steady flow and treated, for the most part, as if it were truly steady, despite continuous minor fluctuations. Our attention will here be devoted to the cases where these mean temporal values continuously vary.

154. Discharge under Varying Head.—A common problem in unsteady flow is that of discharge under a head that continuously changes. Under this condition the rate of discharge will vary, and the problem may be either to find the total volume discharged in a given time interval or to find the length of time necessary for a certain desired result.

Let Q = the total volume in cubic feet of any given body of fluid, while q = cubic feet per second as usual. Then

$$q = \frac{dQ}{dt} \qquad \text{or} \qquad dQ = q\,dt.$$

Suppose that into this body of liquid in question there is an inflow at the rate of q_1 cu. ft. per sec., while liquid flows out at the rate of q_2 cu. ft. per sec. It then follows that the change in the total volume in any time dt is

$$dQ = q_1\,dt - q_2\,dt.$$

Also, let M = the area of the water surface of the body in ques-

tion, while dz = the change in the level of the surface. Then

$$dQ = M\,dz.$$

Equating these two expressions for dQ,

$$M\,dz = q_1\,dt - q_2\,dt. \tag{198}$$

Now, either q_1 or q_2 or both may be variable and functions of z, the variable height of the water surface, or one of the two may have a constant value or be equal to zero. For instance, if water is discharged through an orifice or pipe line of area A under the head z, the following may be written:

$$q_2 = cA\sqrt{2gz},$$

while if it overflows a weir or spillway dam of width b,

$$q_2 = Kbz^{3/2}.$$

In the former case z might correspond to the h in Fig. 154, while the value of c would be determined from the principles of Art. 93 and subsequent articles in case the discharge takes place through a pipe line. In the case of flow over a spillway the z would be the height of the surface of the water above the crest or in other words would correspond to the H in Fig. 98 or of Eq. (72). And in like manner q_1 may also be some function of z.

Equation (198) is perfectly general and if it is possible to express M, q_1, and q_2 as mathematical functions of z, it may then be possible to solve the problem by integration. In other cases the integral may be evaluated by graphical methods. For example, from Eq. (198) may be obtained

$$t = \int_{z_1}^{z_2} \frac{M\,dz}{q_1 - q_2}. \tag{199}$$

By integration this will give us the time required for the water level to change from z_1 to z_2. If it cannot be integrated by calculus, it may be done graphically by computing values of q_1 and q_2 and plotting values of $M/(q_1 - q_2)$ against corresponding values of z. The area between this curve and the z axis is the value of the integral. Of course without actually plotting it, the value of the area may be computed by various rules of approximation.

It may be observed that instantaneous values for the rate of discharge are here determined in the same manner as for steady

flow. This is not strictly correct. For unsteady flow the energy equation is

$$H_1 - H_2 = h_f + h_a, \tag{200}$$

where h_a is the head required for acceleration. In general, it may be either plus or minus in sign. To introduce this into the problem renders the equations much more complicated; and while it does lead to different numerical results in extreme cases, in most cases it will be found to have but negligible effect upon the answer. It seems to be beyond our province to consider it in detail here, but we should bear it in mind.[1]

EXAMPLES

298. Suppose that a ship lock in a canal has vertical sides and that the water discharges through an outlet of area A such that $q = cA\sqrt{2gz}$. Prove that the time required for the water level to fall from z_1 to z_2 is

$$t = \frac{2M}{cA\sqrt{2g}}(z_1^{1/2} - z_2^{1/2}).$$

299. Suppose that water in a storage reservoir whose surface area is constant discharges over a spillway for which $q = Kbz^{3/2}$ and that there is no inflow into the reservoir. Prove that the time required for the height of water on the spillway crest to fall from z_1 to z_2 is $t = \frac{2M}{Kb}\left(\frac{1}{\sqrt{z_2}} - \frac{1}{\sqrt{z_1}}\right)$. How long will it be before the flow ceases?

300. Suppose that a ship lock in a canal is of uniform rectangular cross section and that it is 300 by 90 by 40 ft. deep. Suppose that the water from this lock is discharged through a tunnel which is 3 ft. in diameter, the coefficient of discharge being 0.50. If the initial head under which water discharges is 35 ft., how long will it take for the level to drop 25 ft., that is, from 35 to 10 ft. elevation? *Ans.* 87 min. 30 sec.

301. In Prob. 300 how large would the tunnel have to be to permit the water level to drop from 35 to 10 ft. in 15 min.? *Ans.* 7.27 ft.

302. Water enters a reservoir at a uniform rate of 150 cu. ft. per sec. and flows out over a spillway whose length of crest is 100 ft. The value of K for this spillway is 3.45. Areas of water surface at various elevations above the crest of the spillway are given in the table below. (*a*) Find the time required for the level to drop from 3 to 1 ft. above the crest. (*b*) Find the final elevation after equilibrium is established. (*c*) How long a time will it take for equilibrium to be established?

Ans. (*a*) 2,052 sec. (*b*) 0.573 ft. (*c*) Infinite time theoretically.

[1] PALSGROVE, G. K., and W. J. MORELAND, "Variable Flow of Fluids," *Rensselaer Polytechnic Inst., Eng. and Sci. Series, Bull.* 44.

z, Feet	M, Square Feet
3.00	860,000
2.50	830,000
2.00	720,000
1.50	590,000
1.25	535,000
1.00	480,000

303. The spillway of a reservoir is 40 ft. long and is of such a form that $K = 3.50$. There is a constant inflow into the reservoir of 300 cu. ft. per sec. Areas of water surface are given in the adjoining table. (*a*) What will be the height of the water surface for equilibrium? (*b*) Starting with the water level 3 ft. below the crest of the spillway, how long will it take for the water to rise until the height of the water surface is 1.50 ft. above the crest? *Ans.* (*a*) 1.66 ft. (*b*) 3 hr. 29 min.

z, Feet	M, Square Feet
−3.0	500,000
−2.0	530,000
−1.0	560,000
0.0	600,000
+0.5	650,000
+1.0	700,000
+1.5	740,000

304. The data for a certain flood control reservoir in Southern California are:

Elevation, Feet	Capacity, Acre-feet
1,065	6,557
1,060	5,600
1,054	4,580
1,050	4,000
1,040	2,760
1,030	1,770
1,020	1,000
1,010	450
1,006.5	350
1,000	150
990	40
985	30
969	0

Outlet Works.—The lowest opening is a 24-in. circular gate located with the bottom of the diameter at elevation 985 and discharging into a tunnel 4 ft. square, which the water does not completely fill. The discharge tunnel is served by two 7- by 10-ft. rectangular gates having the longer dimension vertical. The elevation of the sills of the gates is 1,006.5. The

cross-sectional area of the tunnel at the entry end is 172 sq. ft. At a point 108 ft. from the entrance the cross-sectional area is 140 sq. ft., and in the distance of 108 ft. the grade of the tunnel has fallen 19 ft. In an additional 190 ft. of tunnel continuing the cross-sectional area of 140 ft., the discharge end is reached. During this 190 ft. of tunnel length the grade falls 35 ft. This tunnel as well as the spillway channel is of very rough concrete.

Spillway.—The elevation of the spillway floor is 1,054, and the net width of the spillway is 99 ft. 7 in. The grade of the spillway channel is 2.27 per cent.

1. Plot curves of rates of discharge for these three devices against values of elevations as given in the preceding table.

2. What is the maximum rate of discharge possible if the water does not rise above the flood level of 1,065 ft.?

155. Water Hammer.—A different type of unsteady flow from the preceding cases is that of water hammer, which is the phenomenon encountered whenever a flowing liquid in a pipe line is abruptly stopped by the closing of a valve or similar device. This

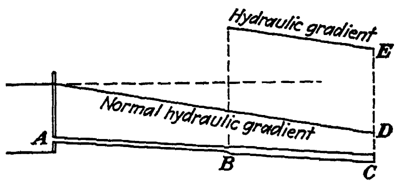


FIG. 201.

will give rise to a series of pressure waves which alternate from plus to minus and may give rise to dangerous pressures.

a. Instantaneous Closure.—We shall first consider an ideal case of instantaneous closure. Referring to Fig. 201, if a valve is abruptly closed at C, the lamina of water next to the valve is compressed by the rest of the column of water flowing up against it. At the same time the walls of the pipe surrounding this lamina will be stretched by the excess pressure. The next laminar layer of water will then be brought to rest by the first, and so on. It is seen that the volume of water in the pipe does not behave as a rigid body but that the phenomena are affected by the elasticity of the water and the pipe. Thus the cessation of flow and the increase of pressure progress along the pipe as a wave action. After a short interval of time the volume of water BC will have been brought to rest, while the water in the length AB will still be flowing with its initial velocity and initial pressure. But the volume of water in BC will be under a much higher pressure due to the compression that it is under, and the pipe walls

will be stretched. The excess pressure DE is the same for all portions of the pipe and is independent of the length of the pipe.

Finally, the pressure wave will have reached the reservoir, and the entire volume of water will be at rest. But a pressure at A higher than that due to the depth of water in the reservoir cannot be maintained, and it instantly drops again to the normal value. A wave of unloading now traverses the pipe from A to C, and by the time that it reaches C the entire pipe is under normal pressure once more. But in the meantime, owing to the compression of the water and the tension of the pipe, the flow has reversed, and water is being returned to the reservoir. The stopping of this reversed velocity now causes the pressure at the valve to drop below the normal value, and thus a wave of rarefaction travels back up the pipe from C to A. This cycle is

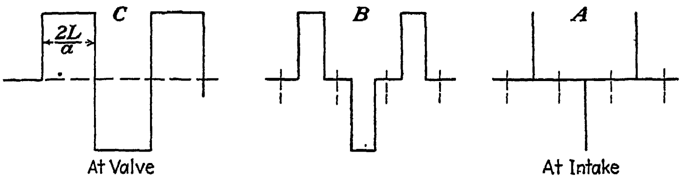


FIG. 202.—Pressure variation with instantaneous closure in ideal case without friction.

repeated over and over again with diminishing amplitude due to frictional effects until it dies away. If the valve were to be alternately opened and closed at just the proper intervals of time, it would be possible to add one pressure wave on top of another, so that there is no limit to the maximum pressure that might be attained. In Fig. 202 is shown the variation of pressure as a function of time for three points in the pipe: at the valve C, at an intermediate point B, and at the intake A. The appearance of the diagram for the intermediate point will approach that of A or C according to the location of B in the pipe line. Figure 202 is purely ideal in that it neglects frictional and other factors, but it serves to give a general idea of the phenomena.

The velocity of a pressure wave in any medium is the velocity of sound in that medium, because sound is transmitted by pressure waves. The velocity of sound is given by

$$a = \sqrt{E_v \frac{g}{w}}, \tag{201}$$

where E_v is the volume modulus of elasticity of the medium in question. Because of the stretching of the pipe walls the apparent diminution of volume under a given increment of pressure is greater than in a rigid pipe. If longitudinal extension of the pipe is prevented by suitable anchors, but circumferential stretching may take place freely, the value of E_v for the liquid in a rigid pipe must be replaced by an equivalent modulus K whose value is

$$\frac{1}{K} = \frac{1}{E_v} + \frac{d}{tE},$$

where d is the diameter of the pipe, t its thickness, and E the tensile modulus of elasticity of the material of the pipe. Substituting this value of K for E_v in Eq. (201) and rearranging, there results an expression for the true velocity of a pressure wave in an elastic pipe:

$$a = \sqrt{E_v\frac{g}{w}}\sqrt{\frac{1}{1 + \frac{E_v d}{Et}}}. \tag{202}$$

For water E_v may be taken as $144 \times 300{,}000$ lb. per sq. ft.; but where the ratio of E_v/E is involved, any units may be employed, as long as they are the same, so E will be taken in pounds per square inch for convenience. The dimensions d and t may also be in any units as long as they are the same. Therefore for water this reduces to

$$a = 4{,}720\sqrt{\frac{1}{1 + \frac{300{,}000}{E}\frac{d}{t}}}. \tag{203}$$

The values of E for steel, cast iron, and wood are about 30,000,000, 15,000,000, and 1,500,000 lb. per sq. in., respectively. The velocity of sound in water or the velocity of a pressure wave in a rigid pipe would be 4,720 ft. per sec., but in elastic pipes of about the usual dimensions it is about 3,300 ft. per sec. In any event it is always less than 4,720.

The time that is required for a pressure wave to travel the entire length of the pipe, which will be denoted by L, is the time required for the entire mass of water to be brought to rest. It is obviously $T = L/a$, and the time for a round trip is

$$T_r = 2\frac{L}{a}. \tag{204}$$

Hence the length of time that a given pressure endures at the valve is $2L/a$. At any intermediate point the time interval is less, as is shown in Fig. 202.

If a volume of liquid is compressed by the application of a pressure p, the average pressure during the process is $p/2$, and the change in volume is equal to p/E_v, but again this modulus of the liquid is to be replaced by an equivalent modulus because of the elasticity of the pipe so that the equivalent compression is equal to p/K. Hence the work done in compressing the liquid and stretching the pipe circumferentially is $p^2/2K$ in foot-pound units, or $p^2/2Kw$ in feet of head. Equating the loss of kinetic energy to the work done on the water and the pipe,

$$\frac{V^2}{2g} = \frac{p^2}{2Kw},$$

from which

$$p = V\sqrt{K\frac{w}{g}}.$$

But since K has replaced E_v in Eq. (201), and

$$\sqrt{\frac{w}{g}} = \frac{w}{g}\sqrt{\frac{g}{w}},$$

it is apparent that the foregoing equation becomes

$$p = w\frac{Va}{g}, \tag{205}$$

where p is the increased pressure in pounds per square foot or similar units due to the water hammer. That is, the magnitude of the pressure wave can be expressed in feet of fluid as

$$y = \frac{p}{w} = \frac{Va}{g}. \tag{206}$$

It will be observed that this pressure increase is independent of the length of the pipe line.

For the sake of clearness in explanation it has been assumed in the preceding discussion that the velocity of the water has

been reduced to zero. But if the valve closure is only partial and not complete, the results are equally true if for V the value ΔV or $V' - V''$ is substituted, where V' is the initial and V'' the final velocity.[1]

b. Rapid Closure.—While a valve may be closed very rapidly, it is physically impossible for the closure to be instantaneous, as assumed in the preceding discussion. Therefore, the case where an appreciable time interval is occupied will be considered. Such a closure may be conceived to be a series of instantaneous movements each one of which has started a pressure wave proportional to the small change of velocity involved. Thus from Eq. (206) should be obtained $dy = (a/g)\, dV$, as the value of the pressure produced by a change in velocity dV. Suppose that a valve movement takes place during a time T_v and that in Fig. 203

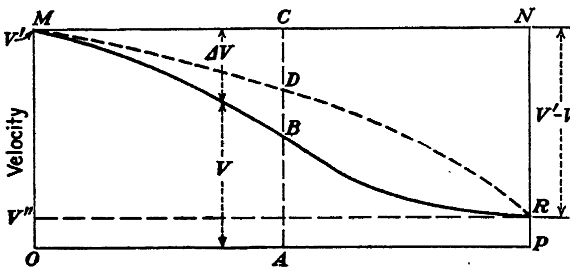


FIG. 203.

there is a representation of the variation in velocity during this time interval, within which the velocity in the pipe decreases from the initial value of V' to the final value of V''. It may be seen that at the end of any time interval, such as OA, the velocity is AB and that the decrease from the initial velocity is BC. Thus the pressure produced at the valve at this instant has attained the value $(a/g)BC$. Hence while ordinates to the curve measured up from OP indicate velocity as a function of time, the ordinates when measured down from MN indicate values of ΔV and thus by a suitable scale will give the time history of the pressure at the valve. It may be noted that the pressure may

[1] The subject of water hammer has been experimentally investigated by Joukovsky of Moscow on pipes of 2-, 4-, 6-, and 24-in. diameter and with lengths ranging from 1,050 to 7,007 ft. He found the results to agree with the formulas given. For a résumé of his work see Miss O. Simin, "Water Hammer," *Trans. Amer. Water Assoc.*, 1904.

vary during this time according to MBR or MDR or any other curve but that its maximum value is proportional to NR or to $(V' - V'')$, the same as if the valve movement had been made instantaneously. If the closure is complete, the value of V'' becomes zero. The essential difference, therefore, between this and instantaneous closure is not in the magnitude of the maximum pressure attained but rather that the vertical lines in Fig. 202 (C) are replaced with curves, such as MBR in Fig. 203, and thus the time during which the maximum pressure is maintained is decreased.

Since any pressure at the valve is transmitted along the pipe with the acoustic velocity a, the curve in Fig. 203 indicates not only the history of the pressure at the valve C but also that at any other portion of the pipe, the distance to which is X, the only difference being a time lag X/a. Furthermore, by the time that the valve movement is completed and the pressure at the valve has reached the maximum value corresponding to NR, the first pressure wave set up will have traveled a distance aT_v. Thus the curve, with ordinates measured down from MN, also indicates the distribution of excess pressure along the pipe for a distance $MN = aT_v$. It may now be seen that with the departure from instantaneous closure, the "vertical front" of the pressure wave in Fig. 201 is replaced with some form of curve.

c. Slow Closure.—The preceding discussion has assumed a closure so rapid, or a pipe line so long, that there was an insufficient length of time for a pressure wave to make the round trip before the valve motion was completed. The time required for a pressure wave to make the trip from the valve to the reservoir and back again is the value given by Eq. (204), that is, $2L/a$. If the valve movement is completed within a time interval less than this, the maximum pressure attained is the same as if it had been made instantly. But if the time interval for the valve movement is greater than $2L/a$, then the maximum pressure attained is less than given by Eq. (206). Thus in Fig. 203 suppose that the time $2L/a$ is equal to OA. Then it may be seen that the wave of pressure "unloading" will reach the valve before the motion is completed, and hence the pressure curve MBR is stopped at the point B. The subsequent pressure history is quite complex, but generally the pressure is not greater than the value corresponding to CB, and it is always less than that represented by NR. But

if the velocity curve had been MDR, the maximum pressure for this case would be a function of CD. Hence as many values of the maximum pressure for slow closure may be found as curves can be drawn between M and R. Thus it is impossible to formulate any simple, general equation for the maximum pressure produced with a slow gate movement. The various formulas that have been proposed, but which are not given in this text, are necessarily restricted to certain special cases and differ in their results because based upon different assumed conditions.[1]

It may be observed that the use of the terms *rapid* and *slow* with reference to the gate movement is purely relative. The criterion is as to whether T_v is less or greater than $2L/a$. In the case of a very short pipe the value of $2L/a$ is so small that it is nearly impossible to close the valve quickly enough to produce water hammer of maximum intensity, while in a very long pipe line care must be taken not to do so. In most practical cases the time of valve movement is greater than $2L/a$.

For any valve closure, which is not instantaneous, the effect of the return "pressure-unloading" wave is to cause the maximum rise in pressure to assume diminishing values as the intake is approached. Thus the hydraulic gradient for the excess pressure, for the actual case, is changed from that in Fig. 201 so that it more nearly resembles the one designated *rejected load* in Fig. 205.

It might be well to add that during a valve closure, the excess pressure produced in the line prevents the rate of discharge, and hence the velocity in the pipe, from diminishing directly with the area of the opening. Thus it is not easy to determine the law of variation of the pipe velocity during this period. It may be seen that the normal head in the pipe also has an influence on this rate of discharge and its variation and is involved in this indirect way. An actual pressure diagram for slow closing is shown in Fig. 204. The maximum pressure is about one-third of the value for instantaneous closure.

Water hammer may be prevented by the use of slow-closing valves, or its effects diminished by the use of automatic relief

[1] For a more adequate treatment of this entire subject see Symposium on Water Hammer, *Amer. Soc. Mech. Eng.*, June, 1933; R. S. Quick, "Comparison and Limitation of Various Water Hammer Theories," *Mech. Eng.*, May, 1927.

valves which permit water to escape when the pressure exceeds a certain value. Also, air chambers of suitable size provide cushions which absorb a great portion of the shock. But for water-power plants a standpipe or surge chamber such as is shown in Fig. 205 has certain marked advantages.

156. Surge Chambers.—Water hammer is characterized by pressure waves which travel with relatively high velocity. While

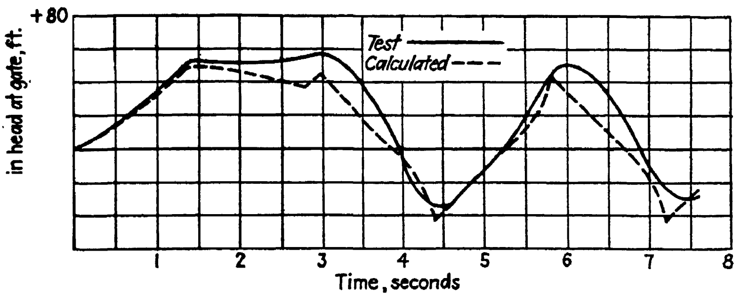


FIG. 204.—Water-hammer studies in Big Creek experimental penstock of the Southern California Edison Company. Conditions: $H_0 = 301.6$ ft., $L = 3{,}060$ ft. of 2.06-in. pipe, $2L/a = 1.40$ sec., $T = 3$ sec., $q_0 = 0.0257$ cu. ft. per sec.

there is a physical movement of the water itself, it is relatively minor. A surge chamber, such as shown in Fig. 205, largely eliminates the pressure waves by permitting the movement of large masses of water. These surges of water are much less rapid than pressure waves, and, while they do cause pressure fluctuations, they are much less severe. They are especially useful in large pumping plants and in water-power plants.

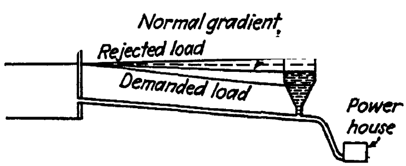


FIG. 205.

In the event of a sudden decrease in load on a water-power plant it would be necessary for the governors rapidly to reduce the amount of water supplied to the wheels, if the speed of the latter is to be maintained constant. A surge chamber provides a place into which this excess water may flow and thus avoids water hammer in the supply pipe. The inertia of the mass of water flowing down this supply pipe may be such as to carry

the water level above the static level and produce an ascending hydraulic gradient. But this excess pressure acts as a retarding force on the mass of water in the pipe line and thus reduces its velocity. In any event, the temporary water level in this surge chamber will be higher than the normal value, and hence it will reduce the velocity of flow too much. This will result in fluctuations of velocity in the pipe line accompanied by "surges" of the water level in the chamber until a condition of equilibrium is finally reached. The phenomenon is very similar to that of water hammer, as there are periodic alternations in pressure and velocity, but the pressure variations are much less severe.

The surge chamber fulfills another valuable function in that it not only takes care of excess water in case of a sudden reduction of flow, but it also provides a source of water supply in the event of a sudden demand. When the load on the plant increases it is necessary to supply more water to the wheels at once. If the pipe line is long, it may take some time to accelerate the entire mass of water, and in the meantime the head at the plant has dropped considerably in order to provide an accelerating force. But the surge chamber permits a certain amount of water to flow out during that period. To be sure, enough flows so that the hydraulic gradient drops below its normal level for the new load, but the effect is not so serious as if the surge chamber were absent.

In Fig. 284 is shown a surge chamber of large size. It is at the end of a pressure tunnel which is approximately 7.76 miles in length, with an average cross-sectional area of 100 sq. ft., and in which is a maximum velocity of flow of 10 ft. per sec.[1]

EXAMPLES

305. A cast-iron pipe line is 24 in. in diameter, and the metal is 0.75 in. thick. If the velocity of water in it is 6 ft. per sec., find the pressure that would be created by the instantaneous closure of a valve.

Ans. 296.5 lb. per sq. in.

306. If the pipe line in the foregoing were 500 ft. long, within what length of time must the valve be closed to produce the same pressure as an instantaneous closure? What would the length of time be if it were 5,000 ft. long?

Ans. $T = 0.27$ sec., 2.7 sec.

[1] DURAND, W. F., "Control of Surges in Water Conduits," *Jour. A. S. M. E.*, June, 1911. See also R. D. JOHNSON, "The Differential Surge Tank," *Trans. A. S. C. E.*, vol. 78, p. 760, 1915.

CHAPTER XIII

DYNAMIC FORCES

157. Dynamic Force Exerted by a Stream.—Whenever the velocity of a stream of fluid is changed either in direction or in magnitude, a force is required. By the law of action and reaction an equal and opposite force is exerted by the fluid upon the body that produces this change. This is called a dynamic force in order to distinguish it from forces due to the hydrostatic pressure.

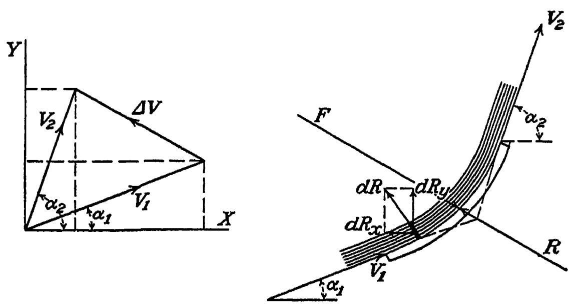


FIG. 206.

First Method.—Let the resultant force exerted by any body upon the water be denoted by R, and its components by R_x and R_y. Let dR be the force exerted upon the elementary mass shown in Fig. 206 whose cross-sectional area is A and whose length along the path is ds. The weight of this elementary volume is $wA\ ds$, and its mass is $wA\ ds/g$. In general, the velocity of a particle of water may vary as a function of both space and time. That is, at a given point the velocity may vary with the time, and it may also change from point to point along its path. Thus, since $V = f(t, s)$, the acceleration is

$$a = \frac{d\bar{V}}{dt} = \frac{\partial \bar{V}}{\partial t} \cdot \frac{dt}{dt} + \frac{\partial \bar{V}}{\partial s} \cdot \frac{ds}{dt};$$

and since $ds/dt = V$, this reduces to

$$a = \frac{\partial \bar{V}}{\partial t} + V\frac{\partial \bar{V}}{\partial s}, \tag{207}$$

where V is a scalar and $\bar{V}$ a vector quantity. For steady flow, to which the present discussion will be restricted, $\partial \bar{V}/\partial t = 0$, and therefore the acceleration is merely

$$a = V\frac{d\bar{V}}{ds}, \tag{208}$$

because the velocity varies along the path but is constant with time at any one point.

Applying to this elementary volume the principle that force equals mass times acceleration, we have

$$d\bar{R} = \frac{wA\ ds}{g}V\frac{d\bar{V}}{ds} = \frac{wAV}{g}\,d\bar{V} = \frac{W}{g}\,d\bar{V},$$

which shows that the dynamic force is proportional to the mass per unit time and to the vector change of velocity. The summation of all such elementary forces along the path at any instant will give the total force exerted. In general, these various elementary forces will not be parallel, and in order that the integration may be an algebraic and not a vector summation, components along the axes will be taken. Thus,

$$R_x = \frac{W}{g}\int_1^2 dV_x = \frac{W}{g}\,V_x\Big]_1^2.$$

Now, at point (1) the value of V_x is $V_1 \cos \alpha_1$, and at (2) it is $V_2 \cos \alpha_2$. Inserting these limits and noting from Fig. 206 that $V_2 \cos \alpha_2 - V_1 \cos \alpha_1 = \Delta V_x$, the result is

$$R_x = \frac{W}{g}(V_2 \cos \alpha_2 - V_1 \cos \alpha_1) = \frac{W}{g}\Delta V_x.$$

Second Method.—The preceding derivation pictures the total dynamic force exerted by a flowing stream to be the vector sum of all the elementary forces exerted along its path at any instant. The following derivation makes it clear that the total force depends solely upon the initial and terminal conditions and is independent of the path. (Of course the numerical value of the terminal velocity would be affected by friction losses which might be different for different paths.) The former method is based on

the principle that resultant force equals mass times acceleration. The second method is based on the principle of force and momentum, which may be stated as follows: The time rate of change of the momentum of any system of particles is equal to the resultant of all external forces acting on the system. Thus instead of $d\bar{R} = md\bar{V}/dt$, it is $d\bar{R} = d(m\bar{V})/dt$.

Consider the portion of a filament of a stream in Fig. 207 that is between two cross sections M and N at the beginning of a time interval dt and between the cross sections M' and N' at the end of the interval. Denote by ds_1 and ds_2 the distances moved during the interval by particles at M and N at the beginning.

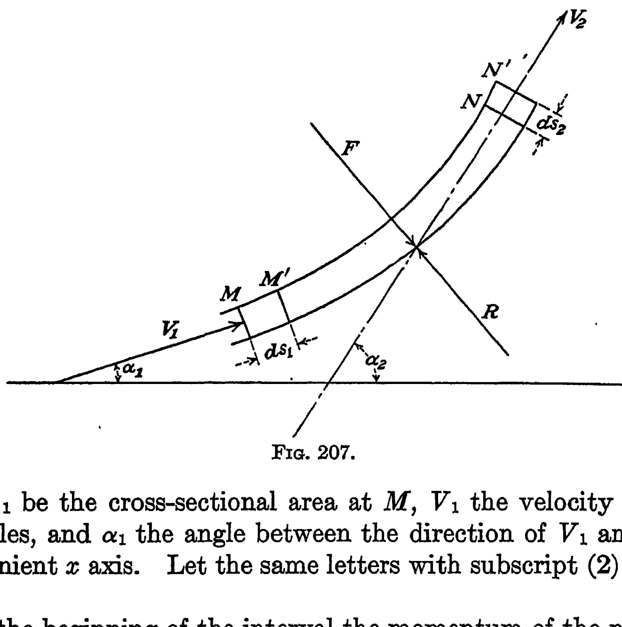


FIG. 207.

Let A_1 be the cross-sectional area at M, V_1 the velocity of the particles, and α_1 the angle between the direction of V_1 and any convenient x axis. Let the same letters with subscript (2) apply to N.

At the beginning of the interval the momentum of the portion of the filament under consideration is the sum of the momentum of the part between M and M' and that of the part between M' and N. At the end of the interval its momentum is the sum of the momentum of the part between M' and N and that of the part between N and N'. In the case of steady flow the momentum of the part between M' and N is constant. Hence the change of momentum is the difference between the momentum of the part between N and N' and that of the part between M and M'.

Noting that $wA_1ds_1 = wA_2ds_2$, since the flow is steady, the change in the x component of the momentum during dt is then

$$\frac{wA_1\,ds_1}{g}(V_2 \cos \alpha_2 - V_1 \cos \alpha_1).$$

If the rate of flow is denoted by W, then

$$wA_1\,ds_1 = W\,dt,$$

and the time rate of change of the x component of the momentum is

$$\frac{W}{g}(V_2 \cos \alpha_2 - V_1 \cos \alpha_1).$$

Denoting by R_x the x component of the resultant force which changes the momentum,

$$R_x = \frac{W}{g}(V_2 \cos \alpha_2 - V_1 \cos \alpha_1) = \frac{W}{g}\Delta V_x,$$

which is identical with the expression derived by the first method.

If F indicates the value of the force exerted by the water, which is equal and opposite to R, then

$$F_x = \frac{W}{g}(V_1 \cos \alpha_1 - V_2 \cos \alpha_2) = -\frac{W}{g}\Delta V_x. \qquad (209)$$

In similar manner the y component of F will be

$$F_y = \frac{W}{g}(V_1 \sin \alpha_1 - V_2 \sin \alpha_2) = -\frac{W}{g}\Delta V_y. \qquad (210)$$

Since

$$F = \sqrt{F_x^2 + F_y^2} \qquad \text{and} \qquad \Delta\bar{V} = \sqrt{\Delta V_x^2 + \Delta V_y^2},$$

the value of the resultant force is

$$F = \frac{W}{g}\Delta\bar{V}. \qquad (211)$$

The direction of R will be the same as that of $\Delta\bar{V}$, and the direction of F will be opposite to it. It is because F and $\Delta\bar{V}$ are in opposite directions that the minus sign appears in the last terms of Eqs. (209) and (210). Note that $\Delta\bar{V}$ is the *vector* difference between V_1 and V_2.

While, in reality, the total force, as shown by the first derivation, is made up of infinitesimal forces along the entire path, it is often convenient to regard it as made up of two single forces concentrated at points (1) and (2), as suggested by the second method. This is analogous to considering a distributed load on a beam, for instance, to be equivalent to one or more concentrated loads. From this point of view, the total force F may be considered equivalent to a force at inflow whose value is $(W/g)V_1$ and whose direction is the same as the velocity V_1 and a second force at outflow whose value is $(W/g)V_2$ and whose direction is opposite to that of the velocity V_2.

158. Dynamic Action upon Stationary Body.—In order to find the dynamic force exerted by a stream upon a stationary object, it is merely necessary to find the value of $\Delta\bar{V}$ in Eq. (211), assuming the rate of discharge to be known. Thus, referring to Fig. 208, let the angle θ be 30 deg. and suppose that the stream is a circular jet with a diameter of 2 in. and a velocity of 100 ft. per sec., while the velocity of the water leaving at (2) is 80 ft. per sec. Taking the x axis as parallel to V_1, then $V_1 \cos \alpha_1 = 100$, and $V_2 \cos 30$ deg. $= 69.3$. Also, $V_1 \sin \alpha_1 = 0$, and

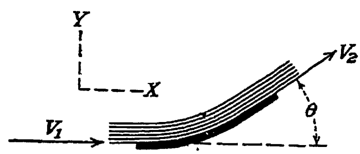


Fig. 208.

$$V_2 \sin \alpha_2 = 40.$$

Since $q = 0.0218 \times 100 = 2.18$ cu. ft. per sec.,

$$\frac{W}{g} = \frac{w \times 2.18}{g} = 4.22.$$

Therefore, $F_x = 4.22 \times (100 - 69.3) = 129.5$ lb., and

$$F_y = 4.22 \times (0 - 40) = -169 \text{ lb.}$$

These two components may then be combined to give

$$F = 212.5 \text{ lb.}$$

In some special cases, a stream may be equally divided in such a way that the value of F_y for one-half will be equal and opposite to that for the other half. Instances of this are where a stream strikes a flat plate normally or where it strikes in the center of a

symmetrical object such as a Pelton bucket. Hence in these cases, which are often met with, $F = F_x$. Also, in the special case where $\theta = 180$ deg., the value of F_y is zero, whether the flow is divided or not.

It may be noted in the foregoing example that the reduction of the velocity at outflow by friction or otherwise increases the value of F_x. But if the angle θ is greater than 90 deg., the reduction of the value of V_2 diminishes the value of F_x. Hence in a Pelton water wheel, where the water is turned through an angle greater than 90 deg., the buckets are polished and made smooth.

EXAMPLES

307. If a jet of water is deflected through an angle θ without any change in the magnitude of the velocity, prove that $F = (2wA/g)V^2 \sin \theta/2$.

308. In Fig. 208 assume that $\theta = 60$ deg. and that the stream striking the body is a jet 2 in. in diameter with a velocity of 100 ft. per sec. If the frictional loss is such as to reduce the velocity of the stream leaving the body to 80 ft. per sec., find (*a*) the component of the force in the same direction as the jet, (*b*) the component of the force normal to the jet, (*c*) the magnitude and direction of the resultant force exerted by the water.

Ans. (*a*) 254 lb., (*b*) 293 lb., (*c*) 388 lb. at 49°08′ with direction of jet.

309. (*a*) Suppose that the jet in Prob. 308 struck a flat plate normally; what would be the value of the force exerted upon the plate?

(*b*) Suppose that the jet were completely reversed in direction or that $\theta = 180$ deg. If V_2 were 100 ft. per sec., what would be the component of the force in the same direction as the jet? What would be the component normal to the direction of the jet?

(*c*) Suppose that the value of V_2 were reduced to 80 ft. per sec. What would be the value of the force exerted?

Ans. (*a*) 423 lb., (*b*) 846 lb., (*c*) 761 lb.

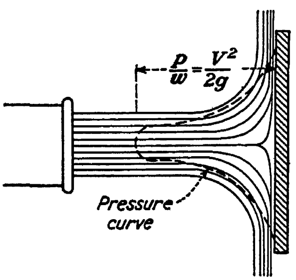


FIG. 209.

159. Pressure on Flat Plate.—As a special case of the preceding, consider Fig. 209 which shows a jet striking a flat plate normally. In accordance with Eq. (211), the total dynamic force exerted is

$$F = \frac{W}{g}\Delta \bar{V} = \frac{wAV}{g}V = wA\frac{V^2}{g}.$$

This is because the velocity component in the direction of V_1 has been reduced to 0. The velocity V_2 is normal to the jet direction and hence can have no component in this direction. Also, since

the fluid leaves the plate symmetrically, the components of V_2 in all directions add up to 0.

If this force is divided by the area of the jet, which is A, the intensity of pressure so obtained would be $p/w = V^2/g$, which is twice as much as it should be, since $V^2/2g = h = p/w$. The fallacy is that the pressure produced by the jet is distributed over an area on the plate that is equal to at least four times that of the jet. If F is divided by 4 A, the result is $p/w = V^2/4g$, which is the average pressure over the plate. The maximum intensity of pressure is the stagnation pressure found in the exact center, and this is $p/w = V^2/2g$.

160. Force Exerted upon Pipe.—When a flowing stream is confined there may be forces due to static pressure as well as dynamic forces due to changes in velocity. Consider the water to be flowing to the right in Fig. 210. Since the velocity is

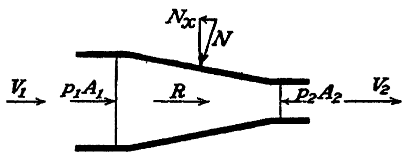


FIG. 210.

increased from V_1 to V_2, the dynamic force exerted upon the water, according to Eq. (211), will be

$$R = \frac{W}{g}(V_2 - V_1).$$

This force, producing the acceleration of the water, must be the resultant of all the forces acting. The real forces acting upon the volume of water shown are the pressures upon the two ends p_1A_1 and p_2A_2 exerted by the rest of the water and the force N exerted by the pipe walls.[1] If there were no friction, this force would be normal to the walls, but actually it will be inclined somewhat from the normal because it must have a frictional component. Let the component of N parallel to the axis of the pipe be denoted by N_x. It may be seen that R must be in the same direction as V_1 and V_2 in Fig. 210. Hence the sum

[1] The N shown in Fig. 210 represents the force for an element only. For a pipe of circular cross section the resultant force exerted by the walls must be axial.

of all the forces parallel to the axis of the pipe must equal R. Therefore,

$$R = p_1A_1 - p_2A_2 - N_x.$$

Inserting the value of R given above, it follows that

$$N_x = p_1A_1 - p_2A_2 - \frac{W}{g}(V_2 - V_1). \qquad (212)$$

It must be remembered that N_x is assumed to be the axial component of the force exerted upon the water by the conical portion of pipe. The force exerted by the water upon the pipe is equal and opposite to this. That is, its magnitude is given by Eq. (212), but it acts toward the right.

If the velocity of the water in a closed passage undergoes a change in its direction, as in the pipe bend shown in Fig. 211, the procedure would be similar to that in the preceding case. The forces acting on the water in the bend are the pressures p_1A_1 and p_2A_2 and the pressure exerted by the walls of the pipe, designated by N. By Eq. (211) the resultant force acting upon this volume of water will be $R = W/g\Delta\bar{V}$, but R is the resultant of the three forces just mentioned. Since these are vector quantities not in the same straight line, it will be better to take x and y components. Thus should be written

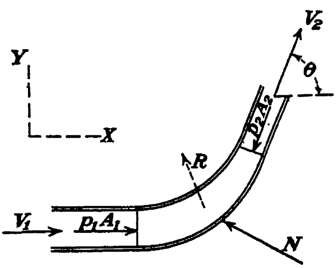


FIG. 211.

$$R_x = \frac{W}{g}(V_2 \cos\theta - V_1) = p_1A_1 - p_2A_2 \cos\theta - N_x$$

and

$$R_y = \frac{W}{g}V_2 \sin\theta = -p_2A_2 \sin\theta + N_y.$$

Solving these equations, it is found that

$$N_x = p_1A_1 - p_2A_2 \cos\theta + \frac{W}{g}(V_1 - V_2 \cos\theta) \qquad (213)$$

and

$$N_y = p_2A_2 \sin\theta + \frac{W}{g}V_2 \sin\theta. \tag{214}$$

But, again, N represents the force exerted by the pipe bend upon the water. The force exerted by the water upon the bend will be equal and opposite to this.

It may be seen that these forces tend to move the portion of pipe considered. Hence a pipe should be "anchored" where such changes in velocity occur.

EXAMPLES

310. On the end of a 6-in. pipe is a nozzle which discharges a jet 2 in. in diameter. The pressure in the pipe is 55 lb. per sq. in., and the pipe velocity is 10 ft. per sec. The jet is discharged into the air. (*a*) What is the resultant force acting on the water within the nozzle? (*b*) What is the axial component of the force exerted on the nozzle?

Ans. (*a*) 304 lb. (*b*) 1,250 lb.

311. Water under a pressure of 40 lb. per sq. in. flows with a velocity of 8 ft. per sec. through a right-angle bend having a uniform diameter of 12 in. (*a*) What is the resultant force acting on the water? (*b*) What is the total force exerted on the bend? *Ans.* (*a*) 137.8 lb. (*b*) 6,530 lb.

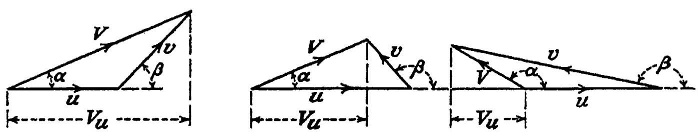


FIG. 212.—Relation between absolute and relative velocities.

161. Relation between Absolute and Relative Velocities.

—In much of the work that follows it will be necessary to deal with both absolute and relative velocities of the water. The absolute velocity of a body is its velocity relative to the earth. The relative velocity of a body is its velocity relative to a second body which may, in turn, be in motion relative to the earth. The absolute velocity of the first body is the vector sum of its velocity relative to the second body and the absolute velocity of the latter. The relation of the three is shown in Fig. 212.[1]

[1] A clearer idea of this relationship may be obtained from the illustration in Fig. 213. Suppose that a raft is moving downstream with a uniform velocity u. A man on the raft at A walks over to the diagonally opposite corner at a uniform rate. But by the time that he reaches B the latter point on the raft will have moved downstream to point C. Thus the path of the man

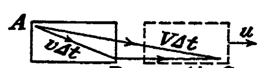


FIG. 213.

162. Dynamic Action upon Moving Body.—The dynamic force exerted by a stream upon a moving object can be determined by a direct application of Eq. (211), provided the flow is steady. The principal difference between action upon a stationary and upon a moving object is that in the latter case it is necessary to deal with both absolute and relative velocities, and the determination of $\Delta\bar{V}$ may be more difficult.

Another point of difference may be in regard to the amount of water that strikes the object. If the cross-sectional area of a stream is A_1, and its velocity is V_1, the rate of discharge is A_1V_1, so that $W = wA_1V_1$. This is the weight of water per unit time that strikes a stationary body.

But in the case of a body having a motion of translation, the weight of water acting may be less than the preceding. As an extreme case, suppose the object to be moving in the same direction as the water and with a velocity equal to, or higher than, that of the stream; it is clear that none of the water will strike it. If the object is moving with a smaller velocity than that of the stream, then the amount of water that strikes it per unit of time will be proportional to the difference between the two velocities. Thus, if W' denotes the pounds of water per second striking a body moving with a velocity u the same direction as the stream, then

$$W' = wA_1(V_1 - u). \tag{215}$$

Our treatment here is necessarily restricted to the case where the body is moving in the same direction as the water, for, if it is otherwise, the case becomes one of unsteady flow, which is quite complex and beyond the scope of this text.

Thus for the case of steady flow, which requires the body to move with a uniform velocity in the same direction as the jet, the force exerted upon a single moving object is

$$F = \frac{W'}{g}\Delta\bar{V}. \tag{216}$$

In general, W' is less than W, and the difference between the two may be accounted for as follows: Suppose that a jet issuing

relative to the raft is AB, but relative to the earth it is AC. Since the velocities are all uniform, they are all proportional to the distances traversed in this interval of time.

from a nozzle strikes an object moving with a velocity u in the same direction. In one second a volume of water A_1V_1 will issue from the nozzle, while only $A_1(V_1 - u)$ will strike the object. But in this same interval of time the object will have moved farther away from the nozzle by a distance u, and thus the volume of water between the two will have been increased by an amount A_1u.

In Fig. 214 is shown a stream of water with a velocity V_1 striking a body moving with a constant velocity u. By the time that a particle of water, which strikes the vane just at the instant it is in the position shown, has reached the point of outflow the vane will have reached the position indicated by the dotted

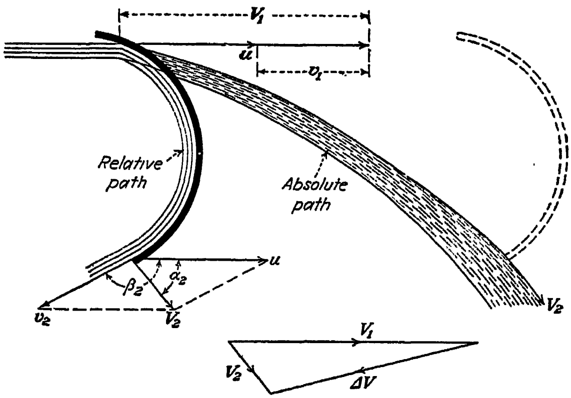


FIG. 214.—Action of fluid on moving vane.

outline. Thus two paths may be traced for the water: one relative to the moving vane, which is as it would appear to an observer who was moving with the vane; and the other relative to the earth, termed the absolute path, as it would appear to an observer standing still with respect to the earth.

A study of Fig. 214 shows that the direction of the relative velocity at outflow from the vane is determined by the shape of the latter but that the relative velocity at entrance, just *before* the water is influenced by the vane, is determined solely by the relation between V_1 and u. Just *after* the water is influenced by the vane, its relative velocity must be tangent to the vane surface. To avoid excess energy loss, these two directions should be made to agree; otherwise there will be an abrupt change in the velocity and path of the water, as is shown in the figure.

As an illustration of the application of our fundamental principles, assume a jet with a diameter of 2 in. and a velocity of 100 ft. per sec. striking an object moving in the same direction with a velocity of 60 ft. per sec. Suppose that $\beta_2 = 150$ deg. and that the friction losses in flow over the vane are such that $v_2 = 0.9v_1$. It is desired to find the force exerted. At entrance to the vane, $v_1 = V_1 - u = 100 - 60 = 40$ ft. per sec. (It must be noted that this is a special relationship because all the velocities are in the same straight line. In general, the relation among the three is as shown in Fig. 212.) At outflow, $v_2 = 0.9v_1 = 0.9 \times 40 = 36$. The quantities V_2 and α_2 may be found by trigonometry, since two sides and an included angle are now known. Thus $\alpha_2 = 32$ deg., and $V_2 = 34$ ft. per sec. It is generally easier to deal with components, however, and, practically, the component of the force in the direction of motion is usually all that is desired. Thus it is found that

$$V_2 \cos \alpha_2 = u + v_2 \cos \beta_2 = 60 - 36 \times 0.866 = 28.8.$$
$$V_2 \sin \alpha_2 = v_2 \sin \beta_2 = 36 \times 0.500 = 18.$$

For a single moving object,

$$W' = w \times 0.0218 \times (100 - 60) = 54.4 \text{ lb. per sec.},$$

and $W'/g = 1.69$. If the x axis is taken in the same direction as u and V_1, then

$$F_x = \frac{W'}{g}(V_1 - V_2 \cos \alpha_2) = 1.69(100 - 28.8) = 120.3 \text{ lb.}$$

and in a similar manner the value of F_y could be found, if desired.

It may be seen that the magnitude of the force exerted by a jet depends upon both the shape and the velocity of the object struck. In fact, the same value of $\Delta\bar{V}$ might be had with either a stationary or a moving object or with moving objects having different velocities, provided only that their shapes, which in this case means their values of β_2, were suitable.

As another illustration of the foregoing, the dynamic force exerted by a jet of water upon the moving body from which it issues may be considered. When a stream of water issues from any device, such as the vessel shown in Fig. 215, a force is required to accelerate the water and impart to it the velocity that it has upon leaving. This force is exerted upon the particles of water

flowing out of the orifice by adjacent particles of water and ultimately by the walls of the vessel. By the law of action and reaction an equal and opposite force will be exerted upon the vessel. It is impossible to analyze this reaction in detail, but its total value will be given by an application of Eq. (211).

Assume that the vessel in Fig. 215 moves to the left with a uniform velocity u and that the orifice is so small compared to the size of the vessel that the relative velocity of the water in the latter may be neglected, as may also the change in h. Then $V_1 = u$. If the jet issues from the orifice with a velocity v_2, the absolute velocity of the jet will be $V_2 = u - v_2$. Hence $\Delta\bar{V} = V_1 - V_2 = u - (u - v_2) = v_2$. Therefore,

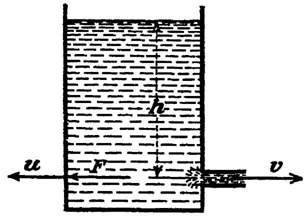


FIG. 215.

$$F = \frac{W}{g}v_2 = \frac{wA}{g}v_2^2. \tag{217}$$

This might have been determined more directly if another proposition had been previously established. That is, that for any case whatever $\Delta V_x = \Delta v_x + \Delta u_x$, where the subscript x merely denotes a component along any axis. In this case, since u is constant, it may be seen that $\Delta V = \Delta v$ and is independent of the velocity of the vessel.

Since $v_2 = c_v\sqrt{2gh}$, then Eq. (217) may be written as

$$F = 2c_v^2 wAh. \tag{218}$$

If losses of energy are neglected in both cases, it may be seen that the reaction of the jet in Fig. 215 is equal to the force of impact upon a flat plate, normal to the jet, provided the area of the plate is large enough to deflect the water through 90 deg.[1]

EXAMPLES

312. A jet of water 3 in. in diameter has a velocity of 120 ft. per sec. It strikes a vane with an angle $\beta_2 = 90$ deg., which moves in the same direction as the jet with a velocity u. Assume that the loss in flow over this vane is

[1] The hydrostatic pressure on an area equal to that of the jet A at a depth h is given by wAh. The fact that this is only half the dynamic pressure considered is of no significance. As has already been pointed out, the dynamic pressure on a plate is distributed over an area much larger than that of the jet, and the intensity of pressure has not increased in either case.

such that $v_2 = 0.9v_1$. When u has values of 0, 40, 60, 80, 100, and 120 ft. per sec., find values of (a) W', (b) $V_2 \cos \alpha_2$, (c) F_x.

Ans. When $u = 40$, (a) 245, (b) 40, (c) 610.

313. If the jet in the preceding problem strikes a vane for which $\beta_2 =$ 180 deg., all other data remaining the same, find values of (a) W', (b) v_2, (c) V_2, (d) F. *Ans.* When $u = 40$, (a) 245, (b) 72, (c) -32, (d) 1,160.

163. Dynamic Action upon Rotating Wheel.—In the case of a water wheel, each bucket or vane may be considered as a body whose motion approximates that of translation for the short distance that it is in the line of action of the jet. Thus the amount of water acting upon a single bucket is W', in accordance with the preceding article. But the wheel as a whole remains in the same position in space and must thus receive the entire rate of discharge W. The explanation is that more than one bucket may be acted upon at the same time, as may be seen in Fig. 239.

Thus to obtain the force acting upon a wheel, as a whole, the value of $\Delta \bar{V}$, or any desired component thereof, may be computed the same as in the preceding article, but it must be multiplied by W/g. Hence, for this case, Eq. (211) is used directly, that is,

$$F = \frac{W}{g}\Delta \bar{V}.$$

In reality, it is physically impossible to have a series of vanes on a rotating wheel moving in a straight line in the same direction as the jet, though such an assumption is often made for the sake of simplicity. Thus, in general, the angle α_1 will not be 0 deg., and the three velocities at entrance will not be as shown in Fig. 214 but rather in one of the forms shown in Fig. 212. This means that the value of W' and also of ΔV for each individual vane will vary with time, and thus the condition of unsteady flow results. However, for the wheel as a whole, the effect upon all the vanes at any instant may be said to give an average which approximates a steady condition. Thus it is permissible to apply our equation to the entire wheel, even under the more general condition.

EXAMPLE

314. Solve Example 313 applying the data there to a wheel instead of one single bucket. *Ans.* When $u = 40$, $F = 1,740$ lb.

164. Torque Exerted.—When a stream flows through a turbine runner in such a way that its distance from the axis of rotation remains unchanged, the dynamic force can be computed by the methods shown. But when the radius varies, it is not feasible to compute a resultant force directly. Instead, it is necessary to find the total torque produced by all the elementary forces. But it has been shown that the total of all the elementary forces may be considered as equivalent to two single forces concentrated at the points of entrance and exit. The torque may then be found by taking the moments of these two forces. Thus, at

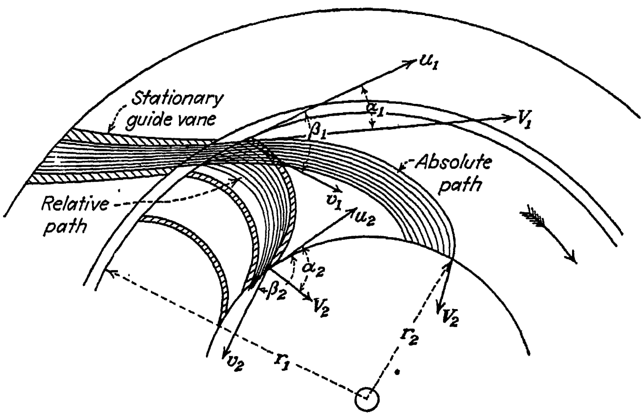


FIG. 216.—Hydraulic turbine.

entrance, the force may be assumed to be $(W/g)V_1$, and at outflow $(W/g)V_2$. Taking moments,

$$T = \frac{W}{g}(r_1V_1 \cos \alpha_1 - r_2V_2 \cos \alpha_2). \tag{219}$$

It is immaterial in the application of this formula whether the water flows radially inward, as in Fig. 216, radially outward, or remains at a constant distance from the axis. In any case, r_1 is the radius at entrance, and r_2 is that at exit.

The solution for the perfectly general case illustrated in Fig. 216 is very similar to the illustrative problem in Art. 162. If the dimensions of the wheel, its speed, and the value of W are given, the procedure would be as follows: The value of V_1 would be determined by the equation $W = wA_1V_1$, where A_1 is the total cross-sectional area of all the guide passages measured normal to

V_1. The angle α_1 is also determined by the position of these guide vanes. The vector triangle at entrance may then be solved by trigonometry for v_1 and β_1, if these quantities are desired. To avoid loss of energy the wheel vane at this point should be tangent to the relative velocity, as determined by the velocity triangle; that is, it should also have the angle β_1. The value of v_2 at outflow may be determined by either one of the two methods that will be given in Art. 169, according to the type of turbine. Then $V_2 \cos \alpha_2$ may readily be computed.

In the case of a centrifugal pump the torque exerted by the impeller upon the water is given by Eq. (219) with the signs reversed. In Fig. 217 is shown the path of water through a pump impeller. If it enters radially, as shown, then $\alpha_1 = 90$ deg.

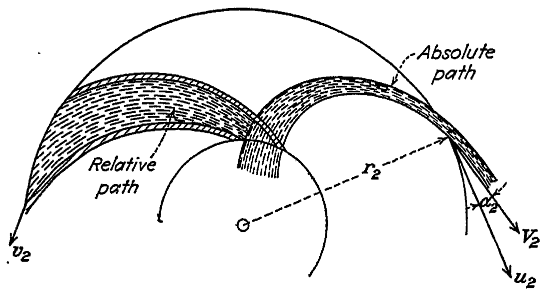


FIG. 217.—Centrifugal pump.

Some pumps have guide vanes within the "eye" of the impeller which impart angular momentum to the water entering the impeller, and for such the value of α_1 is used which is then fixed by construction. But for a pump without these vanes the water may enter the impeller at various angles depending upon the condition of operation. But any rotation of the water in such a case, even though it started back in the suction pipe, has, in reality, been derived from the impeller, for it is obvious that the impeller has supplied directly or indirectly all the angular momentum that the water ultimately acquires. Therefore, for the usual type of centrifugal pump without guides at entrance,

$$T = \left(\frac{W}{g}\right) r_2 V_2 \cos \alpha_2. \tag{220}$$

While Eqs. (219) and (220) are true, they are of little real service, because the proper values to use in them are often not

known with exactness. The precise values of velocities and directions of streamlines are difficult matters to determine. Since water does not fulfill the ideal conditions assumed, it will be found that these equations often yield numerical results that are considerably in error.

165. Power.—Power may be represented by several different combinations, each of which has certain uses, and usually certain of the factors involved may be given various interpretations. Thus the horsepower may be represented by

$$\text{Hp.} = \frac{Fu}{550} \tag{221}$$

$$= \frac{T\omega}{550} = \frac{T \times 2\pi N}{33{,}000} \tag{222}$$

$$= \frac{WH}{550} = \frac{wqH}{550} = \frac{qH}{8.81}. \tag{223}$$

The last form in Eq. (223) applies to water only since it involves the value $w = 62.4$.

In Eq. (221), F must be understood as the component of force in the direction of u. Now, T, for instance, may be the torque measured by a brake, and its use in the foregoing will give us brake horsepower. But if T has the value given by Eq. (219), the power will be greater than the power output of the turbine by an amount equal to the losses in mechanical friction. It will give the power that is actually delivered to the shaft by the water, which is analogous to the indicated horsepower of a steam engine. This is less than the power supplied in the water delivered to the turbine by an amount equal to the power lost in hydraulic friction within the turbine case, runner, and draft tube.

If T has the value given by Eq. (220), the power given by Eq. (222) will be less than that required to run the pump by an amount equal to the mechanical losses, and it will be greater than the power delivered in the water by the amount of the hydraulic losses. It represents the power actually expended by the impeller on the water and is analogous to the indicated power of a reciprocating pump.

In like manner, the quantity H in Eq. (223) may be interpreted as any head for which the corresponding power is desired.

In order to show the application of the preceding, the illustrative problem in Art. 162 may be considered, but the vane is to be

considered as one of a series on a water wheel. Thus W will be dealt with rather than W'. It is seen that

$$W = wA_1V_1 = 62.4 \times 0.0218 \times 100 = 136 \text{ lb. per sec.}$$

Since the velocity is 100 ft. per sec., the head in the jet is

$$H_1 = \frac{V_1^2}{2g} = 155 \text{ ft.}$$

Thus the power in the jet is $136 \times 155/550 = 38.4$ hp. It was shown in Art. 162 that the component of ΔV in the direction of u, which for a wheel means the tangential component, was equal to $100 - 28.8 = 71.2$. Thus the component of the total force in the direction of motion is $F_x = (136/g)71.2 = 300$ lb. The power developed by the water upon the wheel is, therefore, $F_xu/550 = 300 \times 60/550 = 32.8$ hp. The head lost in hydraulic friction in flow over the vanes will be proved later to be represented by $(v_1^2 - v_2^2)/2g = (1{,}600 - 1{,}296)/2g = 4.72$ ft. Thus the power lost in friction is $136 \times 4.72/550 = 1.17$ hp. The kinetic energy discharged from the wheel is represented by $V_2^2/2g = 34^2/2g = 18$ ft., and thus the power lost is

$$136 \times 18/550 = 4.43 \text{ hp.}$$

It is now seen that $38.4 = 32.8 + 1.17 + 4.43$, which checks the calculations.

EXAMPLES

315. Find the power developed for each of the speeds given in Example **312** and using the data there found except for changing the angle β_2 to 180 deg. and considering an entire wheel and not a single vane.

Ans. When $u = 40$, hp. $= 126.7$.

316. The absolute velocity of water entering a turbine runner is 60 ft. per sec., and that leaving is 15 ft. per sec. $\alpha_1 = 20$ deg., $\alpha_2 = 80$ deg., $r_1 = 2.5$ ft., $r_2 = 4.0$ ft.. (*a*) If $W = 600$ lb. per sec., find the torque on the wheel. (*b*) If $u_1 = 50$ ft. per sec., find the power delivered to the wheel.

Ans. (*a*) 2,430 ft.-lb. (*b*) 88.5 hp.

166. Head Utilized.—In turbine and pump practice the word *head* is used to express several different physical quantities. The head h under which a turbine or pump operates is explained in Arts. 126 and 127. But the actual head converted into mechanical work in the turbine is less than this by the amount of the hydraulic friction losses including the energy lost at discharge.

The energy per pound of water that is converted into mechanical work may be called the head utilized and may be designated by h''. Thus the total mechanical work done per second is Wh''. But $Wh'' = T\omega$, where T has the value given by Eq. (219), and h'' replaces H in Eq. (223). Since the angular velocity

$$\omega = \frac{u}{r} = \frac{u_1}{r_1} = \frac{u_2}{r_2},$$

$$(r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)\omega = (u_1 V_1 \cos \alpha_1 - u_2 V_2 \cos \alpha_2).$$

Therefore, $Wh'' = T\omega = (W/g)\ (u_1 V_1 \cos \alpha_1 - u_2 V_2 \cos \alpha_2)$, or the head utilized is

$$h'' = \frac{u_1 V_1 \cos \alpha_1 - u_2 V_2 \cos \alpha_2}{g}. \qquad (224)$$

In the case of the centrifugal pump the mechanical energy that is actually imparted to the water must be greater than the net head h by an amount equal to all the hydraulic losses and may be expressed by the preceding equation with the signs reversed. As has been explained, in many cases the angle α_1 may be considered as 90 deg., so that $h'' = u_2 V_2 \cos \alpha_2/g$, in the case of the usual centrifugal pump.

EXAMPLE

317. Find the head utilized in Example 316. *Ans.* 81 ft.

167. Definitions of Turbine Efficiencies.—The word *efficiency* without any qualifying adjective is always understood to mean gross or total efficiency. It is the ratio of the developed or brake horsepower to the power delivered in the water to the turbine. That is,

$$e = \frac{\text{b.hp.}}{\text{w.hp.}}. \qquad (225)$$

Mechanical efficiency is the ratio between the power delivered by the machine and the power delivered to its shaft by the water. If q represents the total turbine discharge, while q' equals the amount of leakage through the clearance spaces, the actual amount of water doing work is $q - q'$. Hence,

$$e_m = \frac{\text{b.hp.}}{w(q - q')h''/550}. \qquad (226)$$

Hydraulic efficiency is the ratio of the power actually delivered to the shaft to that supplied in the useful water. That is,

$$e_h = \frac{w(q - q')h''}{w(q - q')h} = \frac{h''}{h}. \tag{227}$$

Volumetric efficiency is the ratio of the water actually used by the runner to total amount discharged. Thus

$$e_v = \frac{(q - q')}{q}. \tag{228}$$

The total efficiency is the product of these three separate factors. That is,

$$e = e_m \times e_h \times e_v. \tag{229}$$

168. Definitions of Pump Efficiencies.—The various pump efficiencies are similar to those for the turbine. The total efficiency is

$$e = \frac{\text{w.hp.}}{\text{b.hp.}}. \tag{230}$$

The mechanical efficiency is

$$e_m = \frac{w(q + q')h''/550}{\text{b.hp.}}. \tag{231}$$

The hydraulic efficiency is

$$e_h = \frac{w(q + q')h}{w(q + q')h''} = \frac{h}{h''}. \tag{232}$$

The volumetric efficiency is

$$e_v = \frac{q}{(q + q')}. \tag{233}$$

As in Eq. (229), the total efficiency is the product of these three.

169. Flow through Rotating Channel.—The treatment of the forced vortex (Art. 57) will now be extended to the more general case where the water flows through the rotating vessel. It has been seen that with the free vortex the hydraulic gradient or the resulting Eq. (50) is the same whether the water merely rotates in concentric circles or flows in spiral paths, but with the forced vortex the equation will be found to be somewhat different when

flow occurs. The reason is that in the free vortex no energy, save that lost by friction, is imparted to, or taken from, the water; but in case water flows through a rotating vessel, energy is delivered either to it or by it.

In Art. 166 it was shown that the value of h'' given by Eq. (224) represents head utilized. That is, h'' is the head given up by the water and converted into mechanical work. But if h'' is found to have a negative value, it signifies that energy is being delivered to the water by the vessel instead of being abstracted from it. In practice the former action takes place in a turbine, and the latter in a centrifugal pump.

The general equation of energy may be applied to this case as well as any other, provided that, in addition to the head lost in hydraulic friction, that lost (or gained) in mechanical work is also considered. Hence

$$H_1 - H_2 = h' + h'',$$

where h' represents the head lost in hydraulic friction. This may be expanded by substituting $p/w + z + V^2/2g$ for H and the value given by Eq. (224) for h''. Noting that by trigonometry $V^2 = v^2 + u^2 + 2uv \cos \beta$ and $V \cos \alpha = u + v \cos \beta$, the absolute velocities may be rcplaced by the relative velocities so that the foregoing readily becomes

$$\left(\frac{p_1}{w} + z_1 + \frac{v_1^2 - u_1^2}{2g}\right) - \left(\frac{p_2}{w} + z_2 + \frac{v_2^2 - u_2^2}{2g}\right) = h'. \quad (234)$$

If there is no flow, both v_1 and v_2 become zero, and the equation reduces to that of the forced vortex [Eq. (48)]. If there is no rotation, both u_1 and u_2 become zero, the relative velocity v becomes the same as the absolute velocity V, and the result is the general equation of energy in its usual form.

The head lost in hydraulic friction is proportional to the square of the velocity of flow and is commonly taken as

$$h' = k\frac{v_2^2}{2g}. \quad (235)$$

It may be seen that Eq. (234) is much broader than Bernoulli's theorem in that the latter is only a special case. Its chief use, however, is to fix certain relations between conditions at inflow and outflow from the runner in turbine and centrifugal-pump

theory. Thus where the water in flowing through the wheel is open to the atmosphere, or any constant pressure, $p_1 = p_2$. Usually $z_1 = z_2$ also, and thus the equation reduces to

$$v_1^2 - u_1^2 = v_2^2 - u_2^2 + kv_2^2. \tag{236}$$

And in some special cases, where the flow takes place at a constant distance from the axis of rotation, $u_1 = u_2$, which simplifies the equation still further. In this special case it may be seen that the loss of head in friction is proportional to $v_1^2 - v_2^2$. Equation (236) is usually used to find v_2 when v_1 is known.

In a type of turbine where the water completely fills the entire passages, the various velocities are determined by the equation of continuity, which now takes the form

$$q = AV = av, \tag{237}$$

which may be applied to specific sections, so that

$$q = A_1V_1 = a_1v_1 = a_2v_2 = A_2V_2, \text{ etc.}$$

Note that in each case the cross-sectional area must be measured normal to the direction of the velocity concerned. In this type of turbine, the pressure will not be constant throughout, and Eq. (234) could then be used to find the drop (or gain) in pressure $p_1 - p_2$.

170. Dynamic Force on Submerged Body.—In the first part of this chapter we have been concerned with the action of a jet or stream on a body where the cross-sectional area of the stream is small relative to the body. For example, in the case of a jet acting normally on a flat plate, such as in Art. 159, it is seen that the plate area must be much larger than the jet area in order that all particles of the jet may be fully deflected. Such a case is characterized by the fact that every particle of the fluid may be assumed to be altered in velocity in the same way so that a single value of ΔV applies to each and every particle of the stream. Furthermore, as has been seen in preceding examples, it is possible to calculate ΔV for different shapes of bodies and various other conditions very largely from the geometrical facts involved.

However, where a body is small relative to the cross-sectional area of the stream, it is obvious that not all particles of the fluid are affected in the same way, and hence no single value of ΔV

applies to all of them. We shall here restrict ourselves to the case of a submerged body, where there is no free surface with resulting wave actions to consider, and where also the fluid stream is obviously very large compared to the size of the body. It is also apparent, in theory at least, that the forces exerted by a moving stream upon a stationary body will be the same as those produced by the body moving through a stationary fluid. The disturbance in the velocity field produced by this relative motion extends to very great distances, and therefore a completely satisfactory theoretical treatment becomes more difficult than in the cases previously discussed in this chapter.

The resistance offered to the relative motion of a submerged body due to these dynamic forces is called *drag*. Since we are not here concerned with wave action at a free surface, gravity does not enter into the problem, and the forces involved are those due to inertia and viscosity. The effect of the former is known as *form drag* and is equal to the integration of the components in the direction of the motion of all the normal pressure forces exerted on the surface of the body. It is proportional to V^2 and to the projected area of the body. The effect of the viscosity is known as skin friction or *friction drag* and is equal to the integration of the components in the direction of motion of all the tangential forces exerted over the surface of the body. It is proportional to the fluid viscosity, to V, and to the surface area. The total drag is the sum of these two, and in general it is proportional to some power of V between the first and the second and to an area that is neither the projected nor the total surface area.

As in the case of friction in pipes, the total drag may be represented by an expression of the form $MV + NV^2$, and in certain cases one or the other of these two terms will predominate, depending upon whether the inertia or the friction forces are of primary importance. This may also be put into the forms BV^n or CV^2, and, as in the case of friction in pipes, the coefficient C here involved may be expressed as a function of Reynolds number. When Reynolds number is large, it means that the inertia forces are large compared to the friction forces, since Reynolds number is really the ratio between these two. But even though Reynolds number is large and the viscosity forces apparently negligible, they do exert an important effect, since the pressure forces over the surface depend upon the distortion produced in the velocity

field, and the latter, in turn, is produced through the action of viscosity.

In the case of a flat plate normal to the direction of motion, as in (*a*) of Fig. 218, the tangential forces are all normal to the velocity so that the resistance is purely that of form drag and is due to the difference in the pressures on the two sides of the

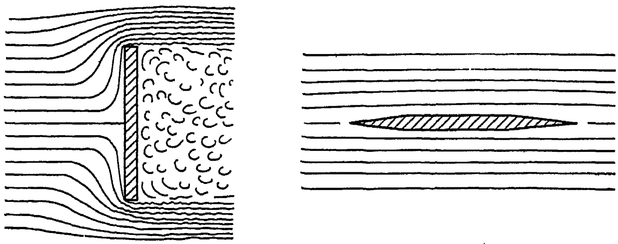


FIG. 218.

plate. This case can clearly be represented by the Newton equation

$$F = C_f \rho A \frac{V^2}{2} = C_f w A \frac{V^2}{2g}, \tag{238}$$

where A is the area of the plate. But, unlike the case in Art. 159, the value of C_f cannot be computed by theory but must be determined experimentally. (For a square plate it has been found to be about 1.1 and to be constant.)

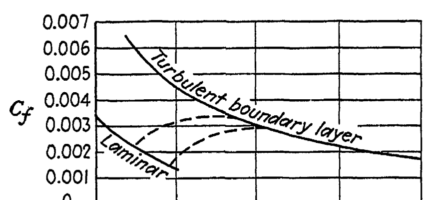


FIG. 219.—Smooth plate parallel to stream.

On the other hand, if a flat plate is placed parallel to the direction of motion, as in (*b*) of Fig. 218, the effect is due almost altogether to skin friction. However, this case is not so simple as the preceding because the value of the coefficient varies as is shown

in Fig. 219, depending largely upon the thickness of boundary layer, and this, in turn, is dependent upon Reynolds number. Also, it is seen that there are two distinct curves depending upon whether the boundary layer is laminar or turbulent. This diagram is seen to be similar in appearance to that for the friction factor in pipes. The linear dimension used for the calculation of

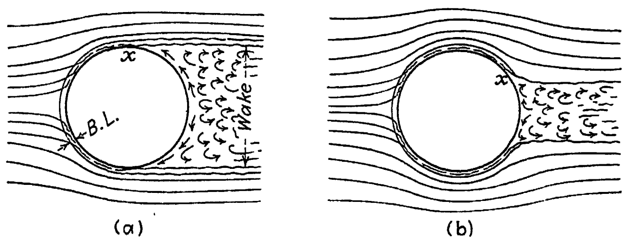


FIG. 220.

values of R for Fig. 219 is that of the length of the plate in the direction of flow.

A case in which both types of resistance occur is that of the sphere shown in Fig. 220, and the variation in its coefficient is shown in Fig. 221. At very low Reynolds numbers, such as $R = 1$ or less, the drag is due to skin friction only, and it may be

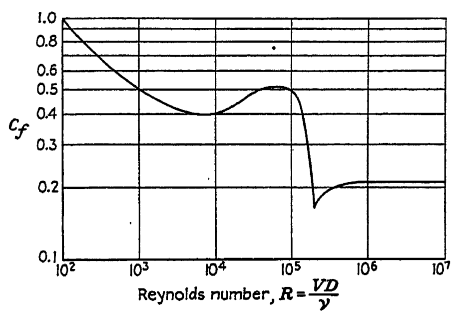


FIG. 221.—Resistance of sphere of diameter D.

shown that Stokes law, which is $F = 6\pi\mu rV$, applies, where r is the radius of the sphere. But this may also be expressed by Eq. (238) with a value of C_f which varies inversely as R in the same way in which friction in laminar flow in pipes may be treated by the V^2 law and a suitable value of f, as is shown in Art. 95. With increasing values of R the inertia forces become more

important, and finally for a range of R from about 10^3 to 10^5 the coefficient becomes nearly constant with values between 0.4 and 0.5. This is because the inertia forces become dominant. However, when R reaches a critical value of about 300,000 there is an abrupt drop in the coefficient, and above $R = 10^6$ it becomes approximately constant again with a value of about 0.2. The explanation of this behavior is that the point of separation of the boundary layer shifts, as may be seen by comparing (*a*) and (*b*) in Fig. 220. The decrease in the width of the turbulent "wake" in case (*b*) diminishes the drag of the sphere.

The boundary-layer theory, as first given by Prandtl, provides a key to the understanding of such cases. The boundary layer has been discussed in connection with the flow in pipes, but a certain amount of repetition here may not be out of order. At large Reynolds numbers, which are usually the cases of engineering interest, there is a thin film next to a boundary wall in which there is a very steep velocity gradient. Therefore the viscous forces $\mu\, dV/dy$ must be large, even though μ be small, as it is in such fluids as water and air. But beyond the thin boundary layer, the ratio of inertia forces to viscous forces is so large that the latter may be disregarded, and the fluid treated as if it were frictionless. Outside this boundary layer, then, the total head H may be considered as constant. Applying the energy equation in this region outside the boundary layer, we can determine variations in pressure caused by the variations in fluid velocity, such as was shown in Art. 51. As long as H may be considered constant, then velocity head can be converted wholly or in part into pressure head, and also the reverse operation can take place. But within the boundary layer conditions are different because of the friction loss caused by the action of viscosity.

If a particle enters the boundary layer near the stagnation point with a low velocity and a high pressure, its velocity will increase as it flows into the lower pressure region along the side of the body, such as a sphere or cylinder, but there will be some retardation due to friction, so that its total useful energy will be reduced by a corresponding conversion into thermal energy. Consequently, when the particle in the boundary layer passes the point of minimum pressure and maximum velocity and begins to decelerate it cannot regain its original pressure and velocity values. Therefore the particle cannot continue on to a rear

stagnation point, as it would do in an ideal frictionless fluid, but will leave the body at a separation point, such as x in Fig. 220. Beyond the separation point there will actually be a reverse flow, as shown. It is this action originating in the boundary layer and made possible by the viscosity that is responsible for the large alteration in the velocity field. This, in turn, produces the form drag, even though viscosity is negligible outside the boundary layer. (If the fluid were frictionless, there could be no form drag due to inertia forces because the pressure distribution on the rear of the body would be similar to that in the front.)

In Fig. 139 it is seen that a boundary layer is, in general, laminar at the start, which means the stagnation point in the bodies here considered, and the boundary layer changes to a turbulent one farther along in the direction of flow. If the Reynolds number for the body is low, the entire boundary layer will be laminar; but if it is sufficiently high, there will be a transition into a turbulent boundary layer. It is an experimental fact that a turbulent boundary layer resists separation more than a

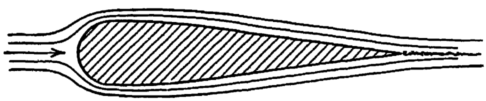


FIG. 222.—Streamline body.

laminar one. This is, in part, due to the fact that shearing stresses are larger in a turbulent flow than in a laminar one, and hence the faster moving outer particles tend more strongly to drag along the slower inner ones and prevent their being slowed down so soon. Hence the separation point is moved farther along, and thus the wake is reduced. Hence we find that although a turbulent boundary layer increases the skin friction, it also reduces the area subjected to low pressure due to the wake and thus reduces the form drag.

A streamline body, such as shown in Fig. 222, is one so shaped that the transformation of velocity head into pressure is so gradual that separation does not occur at all or if it does is confined to a very small region at the extreme end.

In the case of a streamline body the viscosity produces little or no effect, and the form drag is then negligible. Since the resistance to motion is then skin friction, it would seem necessary to express the equation in the form

$$F = C_f \rho S \frac{V^2}{2} = C_f w S \frac{V^2}{2g}, \tag{239}$$

where S is the surface area and not the projected area.

171. Circulation.—The preceding discussion has related to bodies with axes of symmetry parallel to the direction of motion so that in each case the resultant force was also in the direction of motion. We shall now consider cases where there will be a component of force normal to the direction of motion.

Referring to Fig. 223, it is well known that on the underside of the body shown there will be a pressure greater than the pressure in the undisturbed velocity field, while on the upper side there will be a negative pressure. This will result in a force directed vertically upward which we shall designate by L and refer to as the *lift*. The explanation of this pressure distribution

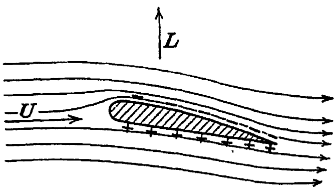


FIG. 223.

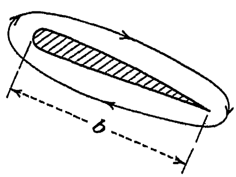


FIG. 224.—Circulation.

is that the velocity on the underside of the body is decreased while that on the upper side is increased, and the observed pressures are then accounted for by the energy equation.

This velocity difference on the two sides could be explained on the assumption that there was superimposed upon the uniform velocity of translation a circulation around the body, as shown in Fig. 224. This velocity would then be added to the velocity of translation on the upper side and subtracted from it on the lower side.

The quantity known as the *circulation* is defined as

$$\Gamma = \oint v \cos \theta \, dl \tag{240}$$

where v is the velocity at a point on the path, and θ the angle between it and the tangent to the path at that point, while l is the length of the path. Equation (240) involves a line integral. It is analogous to the common equation in mechanics for work done as a body moves along a curved path while the force

makes some angle with the path. The only difference mathematically is the substitution of a velocity for a force.

We shall not devote space to the proof for every proposition connected with this subject but merely present it as a useful device which is becoming widely used in the determination of the lift on airplane wings and the forces on turbine and centrifugal pump blades.[1] A fundamental theorem is that the circulation has the same value for every closed curve which completely surrounds the body under consideration. Hence a series of concentric circles will yield the same value as the curve shown in Fig. 224. Since every circle must give the same value regardless of its radius, it follows that vr = constant.

For a circle with radius r, $\theta = 0^0$, and hence $\cos \theta = 1$, while $dl = r\,d\theta$. Thus

$$\Gamma = \oint_0^{2\pi} vr\,d\theta = K\int_0^{2\pi} d\theta = K2\pi. \tag{241}$$

It is furthermore possible to prove that the lift is given by

$$L = \rho b U \Gamma \tag{242}$$

where b is the length of the body perpendicular to the direction of motion or flow and U is the velocity of the body through a stationary fluid or is the velocity of the fluid at a distance from the body if the latter is at rest and the fluid flowing. This equation is restricted to two-dimensional flow. It shows that the lift is directly proportional to the value of the circulation.

In the case of the body shown in Fig. 223, the value of the circulation, and hence of the lift, depends upon both the shape of the profile and the angle of "attack," by which is meant the angle of inclination to U. Its calculation in general is difficult, and therefore the lift is usually determined by experiment, but the concept of circulation is useful in theoretical analyses. Hence, although its numerical value may not often be computed, the term is of fundamental importance.

The lift produced by an airfoil or similar body is caused by the fact that the body is not symmetrical with respect to the fluid stream, and hence the velocity field around it is not symmetrical. In the case of flow past a cylinder, whose axis is perpendicular to

[1] For details see PRANDTL-TIETJENS, "Applied Hydro- and Aeromechanics," p. 160, McGraw-Hill Book Company, Inc., New York, 1934.

the flow, there is no lift because of symmetry. However, if the cylinder is rotated about its axis, it will tend to carry fluid around with it and thus produce a distortion in the velocity field.

With an airfoil there is no physical circulation of fluid, as this is only a mathematical concept in such a case. But with a rotating cylinder there is always a tendency for actual fluid to be carried around; and if the velocity of rotation is high enough relative to the uniform velocity U, a ring of fluid may actually be dragged around.

At the surface of the cylinder, if no slip is assumed, the circulation is therefore

$$\Gamma = 2\pi r_1 u_1 = 2\pi r_1^2 \omega,$$

where r_1 is the radius of the cylinder, u_1 the peripheral velocity, and ω its angular velocity. Inserting this value for circulation in Eq. (242), the lift per unit length of the cylinder is

$$L = \rho U 2\pi r_1 u_1 = \rho U 2\pi r_1^2 \omega.$$

If $u_1 = mU$, then also

$$L = 4\pi m \rho r_1 \frac{U^2}{2} = 4\pi m w r_1 \frac{U^2}{2g}.$$

This same result can be obtained, somewhat more laboriously, by determining the velocity variation around the circumference; computing the pressure distribution from it by means of the energy equation, assuming no friction loss (Bernoulli's theorem); and then integrating over the surface. In Art. 46 was shown the method of determining the velocity field around a body and the variation of velocity along its surface. The bodies there dealt with were infinitely long, so that we were concerned with the "source" but not with the "sink." If the sink and the source are brought closer together until they coincide, the case is known as a *doublet*, and the body becomes a sphere in three-dimensional flow or a cylinder in two-dimensional flow. In the latter case it is possible by the aid of potential functions, which will not be discussed in this text, to obtain an equation for the velocity around the circle.

If θ is the angle between any radius of the circle and the direction of the uniform velocity U, while V is the velocity at the surface of the cylinder, then, from potential functions,

$$V = 2U \sin \theta.$$

From this it is seen that the velocity at the stagnation point at 0° is zero, while the velocity on either side of the circle where $\theta = \pm 90°$ is $2U$. If the velocity u_1, due to the circulation, is now included, the resultant velocity is $2U + u_1$ on one side and $2U - u_1$ on the opposite side. Thus the pressure will be decreased on the former side and increased on the latter.

The general equation for the pressure at any point on the circumference is obtained as follows:

$$\frac{p}{w} + \frac{V_c^2}{2g} = \frac{p_0}{w} + \frac{U^2}{2g},$$

where p_0 is the pressure at some distance away where the velocity is uniform, and

$$V_c = 2U \sin \theta + u_1$$

is the resultant velocity at any point on the surface. From these two equations

$$p - p_0 = \frac{\rho}{2}(U^2 - 4U^2 \sin^2 \theta - 4U \sin \theta u_1 - u_1^2).$$

Since the elementary area per unit of length of the cylinder is $r_1\, d\theta$, and the lift is the summation of all the components normal to the direction of U, the resulting value of L is obtained from

$$L = \int_0^{2\pi} (p - p_0) r_1 \sin \theta \, d\theta.$$

Substituting the expression for $p - p_0$ and integrating, this reduces to

$$L = 4\pi m \rho r_1 \frac{U^2}{2},$$

as was obtained previously. This latter procedure has been presented here to demonstrate the validity of Eq. (242) and also to show how much detailed labor is avoided if the numerical value of the circulation Γ (gamma) is known, so that Eq. (242) can be employed.

Of course this result is purely ideal, since the fluid will not rotate at the same speed as the cylinder. A lift coefficient of between 9 and 10 has been obtained, as compared with the 4π

shown in the equation, but the cylinder had to be rotated much more rapidly than the value indicated in order to bring the fluid up to the desired speed. This large slippage between fluid and cylinder results in the expenditure of much power in fluid friction, and hence the Flettner rotor, employing this principle, is not a successful device.[1]

172. PROBLEMS

318. If a jet of water strikes a body moving in the same direction and flows over it without friction loss, prove that $F_x = (wA_1/g)(1 - \cos\beta_2)(V_1 - u)^2$. If it acts upon a wheel under similar conditions, prove that $F_x = (wA_1/g)(1 - \cos\beta_2)V_1(V_1 - u)$.

319. Using the latter expression in the preceding problem for the force exerted upon a wheel, prove that the power developed is a maximum when $u = 0.5V_1$.

320. A jet of water with an area of 3 sq. in. and a velocity of 100 ft. per sec. strikes a stationary vane which deflects it through an angle of 135 deg. The loss in flow over the vane is such that $v_2 = 0.8v_1$. Find the components of the force exerted in the direction of the jet and at right angles to it. Find the forces exerted if there is no friction loss.

321. Solve Prob. 320 for both cases, if the angle of deflection is 45 deg.

322. Suppose that the vane in Prob. 320 is moved towards the nozzle, from which the jet issues, at a speed of 30 ft. per sec. Find the components of the force exerted. *Ans.* $F_x = 1{,}067$ lb., $F_y = 386$ lb.

323. Solve the above if the velocity of the vane is 30 ft. per sec. away from the nozzle.

324. A jet of water issuing from an orifice in a vessel under a head h_1 strikes a large flat plate which covers the end of a tube in a second vessel, in which the height of water above the tube is h_2. The area of the tube is equal to that of the jet. If the impact of the water is just sufficient to hold the plate in place, neglecting its weight, prove that $h_2 = h_1$.

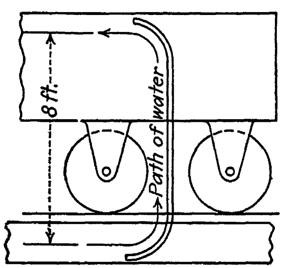


FIG. 225.

325. A familiar type of lawn sprinkler, which is a true reaction turbine, consists of two or more horizontal arms which rotate about an axis due to the reaction of jets issuing from orifices. If there are two arms, each with an orifice $\frac{1}{4}$ in. in diameter at a distance of 14 in. from the axis and so placed that the jets issue at right angles to the radius, find the torque exerted if the pressure of the water within the tube is 60 lb. per sq. in., assuming the coefficient of discharge to be unity.

326. A locomotive tender running at 20 m.p.h. scoops up water from a trough between the rails, as shown in Fig. 225. The scoop delivers the water

[1] For a much more extensive treatment of circulation, see W. F. Durand, "Aerodynamic Theory," vol. I, p. 154.

8 ft. above its original level and in the direction of motion. The area of the jet of water at entrance to the scoop is 50 sq. in. The water is everywhere under atmospheric pressure. Neglecting all losses, what is the absolute velocity of the water as it leaves the scoop? What is the force acting on the tender due to the water? What is the minimum speed of the train at which water will be delivered to the tender?

(NOTE.—The equation of relative velocities may be applied with $u_1 = u_2$, as it must be with translation.)

327. Find the horsepower of a jet of water with a cross-sectional area of 3 sq. in. if it has a velocity of 100 ft. per sec. What is the force of reaction?
Ans. 36.8 hp.

328. Suppose that the jet in Prob. 327 were to strike a wheel with curved vanes. Assume that $\alpha_1 = 0$ deg., $r_1 = r_2$, and that the vanes reversed the relative velocity of the water through 180 deg. without friction loss. Find values of the force exerted when the peripheral speeds of the vanes are 0, 30, 50, 80, and 100 ft. per sec.
Ans. 808, 566, 404, 161.5, and 0 lb., respectively.

329. Find the horsepower and efficiencies for the five speeds given in Prob. 328. *Ans.* 0, 30.8, 36.8, 23.5, and 0 hp., respectively.

330. Suppose that the wheel in Prob. 328 were equipped with vanes for which $\beta_2 = 160$ deg. and that the loss in flow over the vanes was such that $v_2 = 0.8v_1$. Find the values of force exerted, power, and efficiency for the five speeds given.

331. A water wheel is used under a net head of 450 ft. and runs at 350 r.p.m. The rate of discharge is 138 cu. ft. per sec. The angle $\alpha_1 = 15$ deg., $\beta_2 = 162$ deg., $r_1 = 2.0$ ft., $r_2 = 2.5$ ft., and $k = 0.2$. The wheel is of the "impulse" type, so that $p_1 = p_2$. Let $z_1 = z_2$. Find v_1, v_2, V_2, α_2, torque, head utilized, hydraulic efficiency, and power delivered by water. *Ans.* $v_1 = 101$, $v_2 = 105$, $V_2 = 33.4$, $\alpha_2 = 103$ deg. 40 min., $T = 93{,}300$, $h'' = 396$ ft., $e_h = 0.88$, hp. $= 6{,}220$.

332. An impulse turbine has the following dimensions: $\alpha_1 = 20$ deg., $\beta_2 = 160$ deg., $r_1 = 1.5$ ft., $r_2 = 1.8$ ft., $k = 0.5$. The net head supplied is 300 ft., and the rate of discharge 57 cu. ft. per sec. If $u_1 = 63.5$, it will be found that $v_2 = u_2$. For this speed find the head utilized, the torque, hydraulic efficiency, and power delivered to the vanes.
Ans. $h'' = 247$ ft., 1,600 hp.

333. A turbine operating under a head of 50 ft. discharges 30 cu. ft. of water per sec. when running at 660 r.p.m. It is of the "reaction" type so that the water completely fills the runner passages under a pressure that varies along the path. Let $z_1 = z_2$ and assume that leakage is negligible. The dimensions are $\alpha_1 = 35°$, $\beta_2 = 155°$, $r_1 = 0.70$ ft., $r_2 = 0.42$ ft., $A_1 = 0.837$ sq. ft., $a_2 = 0.882$ sq. ft. Find magnitude and direction of relative velocity at entrance to the runner and of the absolute velocity at exit from the runner. *Ans.* $v_1 = 28$, $\beta_1 = 133°$, $V_2 = 14.53$, $\alpha_2 = 97°$.

334. In Prob. 333 find the torque exerted upon the shaft and the horsepower delivered to the shaft. *Ans.* 1,238 ft. lb., 155.5 hp.

335. In Prob. 333 find the head utilized by the turbine, the hydraulic efficiency, and the brake horsepower if the mechanical efficiency is 96 per cent. *Ans.* $h'' = 45.7$ ft., $e_h = 0.914$, 149 b.hp.

336. If $k = 0.05$ in Prob. 333, find the drop in pressure between entrance to and exit from the runner. *Ans.* 29.9 ft.

337. In Prob. 333 find the head lost within the runner due to friction, assuming $k = 0.05$, and the energy loss at exit from the runner.

Ans. 0.9 ft., 3.3 ft.

338. If the turbine in Prob. 333 is held so that the runner cannot rotate, the discharge will be found to be 29.5 cu. ft. per sec. What will be the torque exerted on the shaft? *Ans.* 1,890 ft.-lb.

CHAPTER XIV

DESCRIPTION OF THE IMPULSE WHEEL

173. Impulse and Reaction of a Jet.—When a stream of water strikes any object, the dynamic force exerted, due to the impact, is often termed the *impulse* of the jet. The dynamic force exerted by the jet upon the vessel from which it issues is often called the *reaction* of the jet. But in both cases the force is due to the change that is produced in the velocity of the water.

174. Distinction between Impulse and Reaction Turbine.—The distinction between these two fundamental types of turbines, according to the action of the water as defined in the preceding article, was proper in primitive wheels. But in modern turbines the so-called impulse at entrance and reaction at exit may both be effective in either type. A better classification is as to *pressureless* and *pressure* turbines.

Thus the water within the impulse wheel is not confined but is open to the air, while in the reaction turbine the wheel passages must be completely filled with water. In the former the pressure remains unchanged in flowing over the buckets, while with the latter the pressure decreases during flow through the runner. The energy delivered to the impulse turbine is all kinetic, while that delivered to the reaction turbine is partly kinetic and partly "pressure energy."

But it is well to bear in mind that in both types the essential thing is that the velocity of the water must be altered in order that a dynamic force may be exerted upon the wheel. And in both types it is necessary, if high efficiency is attained, that the absolute velocity of the water as it leaves the wheel be low, since this velocity represents so much kinetic energy that is not utilized.

175. The Impulse Wheel.—There have been several types of impulse turbines produced, but the only one that has survived in this country is of the kind shown in Fig. 226. This is the impulse wheel or the Pelton wheel, so called in honor of L. A. Pelton who contributed to its early development. It may be also designated by the name of the tangential water wheel, from

FIG. 226.—Impulse water wheel with needle nozzle. (Needle is drawn back and nozzle is wide open.)

FIG. 227.—Impulse wheel viewed from below. (Nozzle closed by needle.) $D = 84''$; $h = 134'$; $N = 124$; hp. $= 280$.

the fact that the center line of the jet is tangent to the path of the center of the buckets.

The wheel in Fig. 226 is operated by a jet of water from the nozzle at the left. This same wheel in action may be seen in Fig. 241. A view of another wheel showing the relation of the nozzle to the buckets is shown in Fig. 227. The jet strikes the dividing ridge, or *splitter*, of the buckets, is divided into two

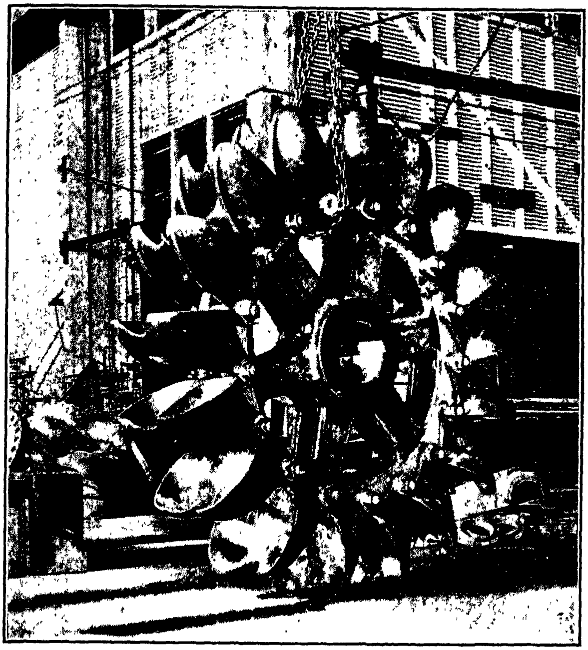


FIG. 228.—Pelton wheel, in shop of Pelton Water Wheel Co. $D \doteq 76''$; $h = 540'$; $N = 257$; hp. $= 2{,}100$.

parts, flows over the face of the bucket, and is finally discharged at both sides of the latter.

In Fig. 228 may be seen a view of an assembled wheel with the "chain type" of construction. That is, each bolt is instrumental in holding two buckets, so that the latter are fastened together as a chain. This permits of a compact construction and enables the buckets to be placed closer together than in the type shown in Fig. 226.

The device shown at the right in Fig. 226 is the *stripper*, its function being to prevent water being carried around with the

wheel and thus adding to the windage losses. The buckets pass through an opening in this with a clearance of about 0.5 in.

Since the generator rotor is usually heavier than the Pelton wheel, it is usually mounted between two bearings, and the Pelton wheel is installed on the projecting end of the shaft beyond one of the bearings. This is known as the *single overhung construction.*

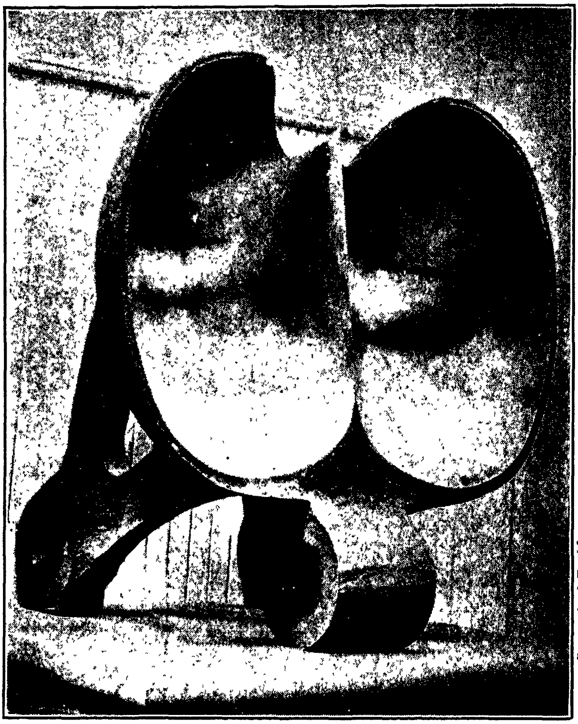


FIG. 229.—Pelton ellipsoidal bucket.

Often two wheels drive one generator between them, and in this case each wheel is mounted on the end of the shaft, as shown in Fig. 238. This is known as the *double overhung construction.*

176. Buckets.—Typical styles of buckets now in use for impulse wheels are seen in Figs. 229 and 230. The theory shows that the face of the bucket should be a surface of double curvature, and it is also found that the shape of the back of the bucket may be as important as that of the face. The reason for this is that the back of the bucket may interfere with the water which is

acting upon the bucket ahead, for when a bucket swings down into the jet it merely cuts off the jet from the preceding bucket

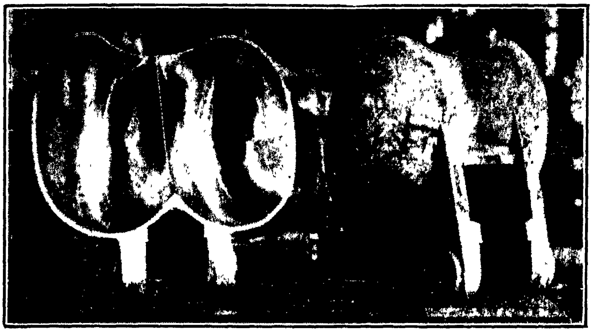


Fig. 230.—Allis-Chalmers buckets. (*Courtesy of Allis-Chalmers Mfg. Co.*)

and leaves a "slug" of water to complete its work on the one ahead. If the back of the bucket is not properly shaped, it may

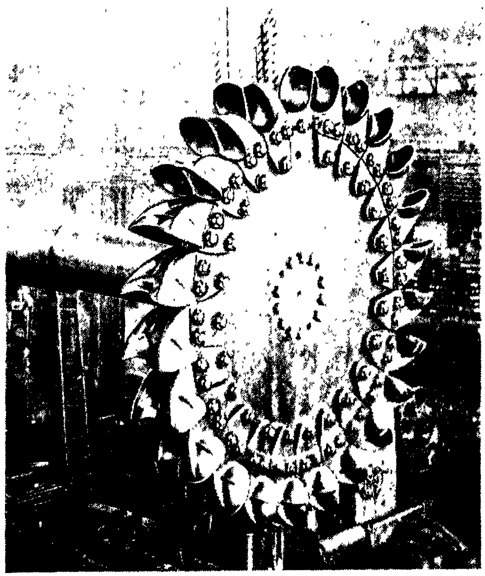


Fig. 231.—Impulse wheel at Big Creek 2A plant of the Southern California Edison Company, 56,000 hp. (65,000 max.), 2,200 ft. head, 250 r.p.m. (*Courtesy of Allis-Chalmers Mfg. Co.*)

not leave sufficient clearance for the water. The "notch" is cut out of the Pelton bucket so that it may reach a position where

its path is more nearly tangent to that of the jet before the latter strikes it.[1]

For service under moderate heads these buckets may be made of cast iron, though the better ones are of bronze or steel. For very high heads only the latter may be employed. The working face of the bucket should be smoothed up or polished, and the dividing edge, or splitter, ground to a knife-edge in order to reduce hydraulic friction losses.

For high efficiency it is desirable that the bucket reverse the relative velocity of the jet as nearly as is feasible. But a com-

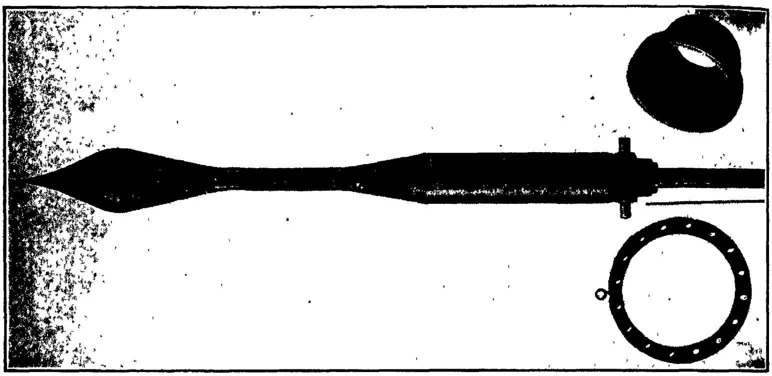


Fig. 232.—Pelton needle and nozzle tip. (*Courtesy of Pelton Water Wheel Co.*)

plete reversal of 180 deg. is not permissible, as the water must be thrown to one side so as to clear the following bucket. An angle of about 165 deg. is usually employed, though even 170 deg. may frequently be used. Because of surface tension the actual direction of the water will always be somewhat less than the bucket angle, the difference between the two decreasing as higher heads are used. For good efficiency the width of the bucket should be at least three times the diameter of the jet, and the diameter of the wheel should be at least nine times that of the jet. (The usual ratio is 12 in the latter case.) Since jets 10 in. or more in diameter are in use, buckets of at least 30 in. in width are sometimes seen.

[1] For impulse wheels of high specific speeds there are other reasons for this construction which space forbids taking up here in detail. In brief, it is so that every bit of water may complete its work upon the bucket before the latter leaves the line of action of the jet, in which event some of the water would not be utilized (see Fig. 239).

177. Nozzles and Governing.—The jets used in impulse wheels are almost always furnished by needle nozzles such as shown in Figs. 233 and 234. The needle itself is shown in Fig. 232. As it is moved back and forth in the nozzle, it varies the size of the nozzle opening and hence varies the amount of water discharged. But fortunately it does not involve any serious loss of head until the nozzle is nearly closed. The efficiency of a needle nozzle when it is wide open may be about 97 or 98 per cent, the velocity coefficient being about 0.99 or a little less. The nozzle efficiency would not fall below 90 per cent until the needle was closed so far

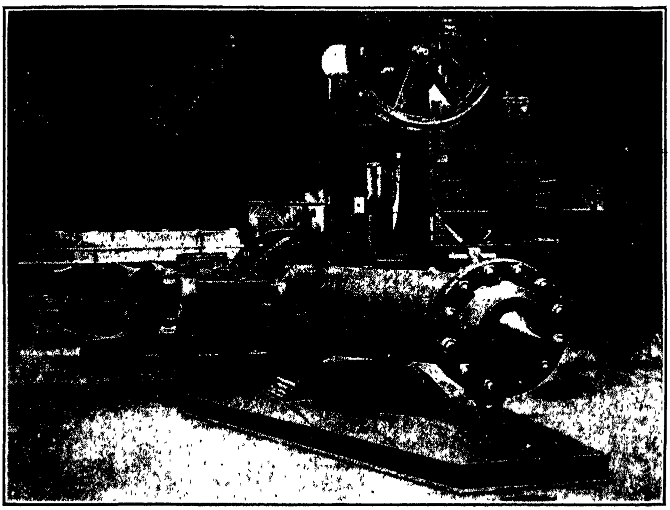


FIG. 233.—Deflecting needle nozzle for a 10,000-hp. jet.

that about half the maximum amount of water was being discharged. Thus it is a very efficient regulating device.

In order to keep the speed of a wheel constant under different loads it is necessary to vary the amount of water so that the power supplied to the turbine will be proportional to the power demanded. This can sometimes be done by changing the position of the needle in accordance with the power that the wheel must deliver. Under certain conditions the governor may control the position of the needle for this purpose. But if the changes of load are rapid, and the pipe line is long, this procedure would involve serious water hammer, if close speed regulation were attempted.

In order to secure close speed regulation and yet be free from the danger of water hammer, the deflecting nozzle is often used. The entire nozzle is movable about a ball-and-socket joint near the base and swings on trunnions. In case of a sudden drop of load on the machine, the governor could lower the end of the

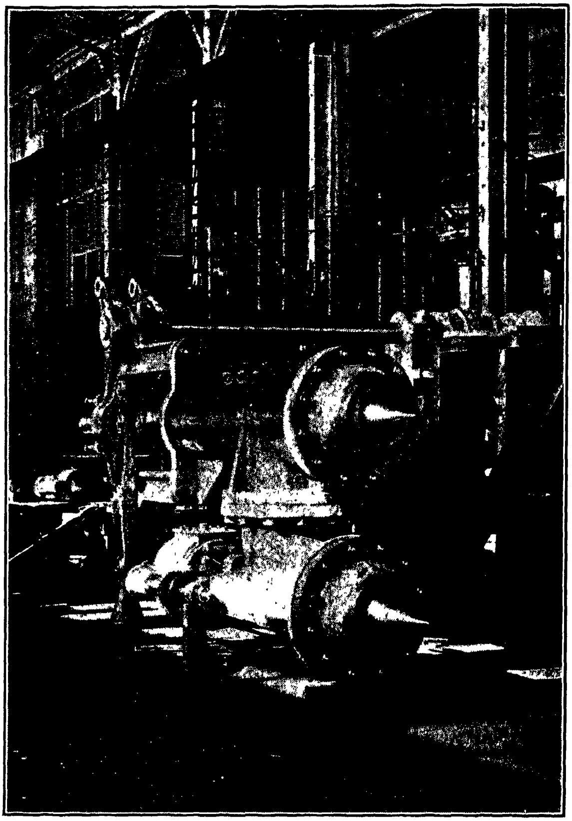


FIG. 234.—The needle nozzle with auxiliary relief.

nozzle so that only a small part of the jet struck the buckets, the rest of the water being wasted. As the load increased, the nozzle could be raised so that a larger amount of water would strike the wheel. As this would be wasteful of water, such nozzles are almost always equipped with needles as well, which can be set by the station attendant in accordance with the load that the wheel carries. Thus water would be wasted for a short time only, but the needle would be closed so slowly that no damage

would be done to the pipe line. But the nozzle may be deflected with any degree of rapidity so that close speed regulation may be secured. Of course, in case of an increase in load it would be necessary for the operator to open the nozzle, as the governor is powerless there. But the experience is that increases of load come on gradually enough for this to be done. The chief function of the governor is to prevent racing in cases of abrupt decreases

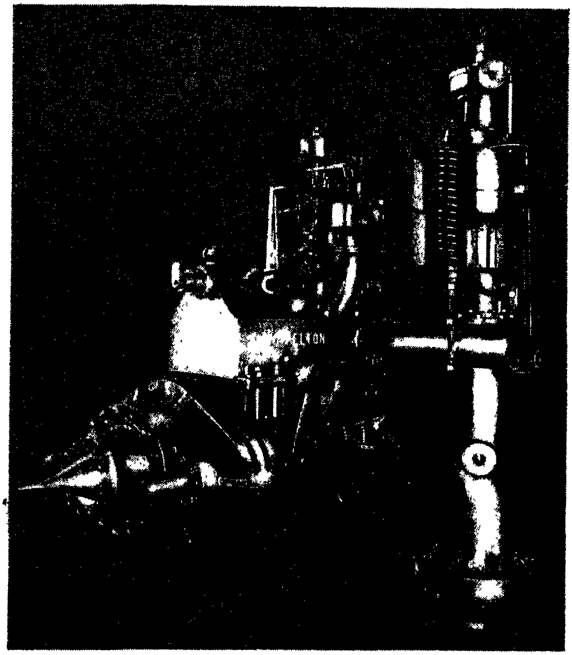


FIG. 235.—Needle nozzle for 18,000-hp. wheel under head of 1,280 ft. (*Courtesy of Pelton Water Wheel Co.*)

in load. Occasionally, the nozzle is so made that the governor deflects it first and then slowly closes the needle.

The needle nozzle with an auxiliary relief, as shown in Fig. 234, is frequently used. In this type the jet from the upper nozzle strikes the wheel, while that from the lower nozzle goes below it. It is so arranged that when the governor closes the upper nozzle it opens the lower one. Thus there is no abrupt change in flow in the pipe line, as the surplus water simply flows out through another place. But in order to prevent waste of water, the connection between the governor and the auxiliary nozzle is a dash-

pot arrangement which permits the needle to be moved only when the governor movement is rapid, and when the relief has been opened this arrangement permits it to be gradually closed again. Thus close speed regulation has been accomplished, and also economy in the use of water has been secured.

FIG. 236.—DeSabla power plant in normal operation. Under head of 1,531 ft. (*From a photograph by F. H. Fowler.*)

The nozzle shown in Fig. 235 operates in a similar fashion, except that the water is by-passed through a relief valve shown in the lower right-hand corner. Another arrangement is to have a deflector out in front of the nozzle which diverts some of the water from the wheel until the needle valve can be adjusted to accommodate the decreased load. This also permits rapid governing action without danger of water hammer and with very little waste of water.

It will be noted that all of these devices may prevent rapid changes in flow in the pipe line in the case of decreasing loads. But only a surge chamber located near the wheels will be able to supply water in the case of a sudden demand.

178. Conditions of Service.—The impulse wheel is particularly adapted for service under high heads, though it may also be

FIG. 237.—DeSabla power plant with nozzles deflected. (*From a photograph by F. H. Fowler.*)

employed under low heads if the power is small. In fact, the choice of the type of turbine is a function of power as well as of head.

The highest head that has ever been developed is one of 5,800 ft. in Canton Valais, Switzerland. This is said to deliver 30,000 hp. in a single jet, which is very nearly the record for power as well.

For a number of years there has been a plant in Switzerland operating under a static head of 5,412 or 4,850 ft. net at the nozzles. There are five wheels of 3,000 hp. each running at

500 r.p.m. The diameter of each wheel is 140 in., and that of the jet is 1.5 in.

In the United States the highest head that has been used is 2,562 ft. static with a net head of 2,350 ft. at the nozzle. This is in a plant of the Feather River Power Company in northern California. There are a number of plants in operation under heads of about 2,000 ft.

The jets used upon Pelton wheels are of all sizes up to a maximum, so far, of 14 in. The latter is in one of the units of the Bureau of Water and Power of the City of Los Angeles. Ordinarily, only one jet is used upon a single wheel, but occasionally

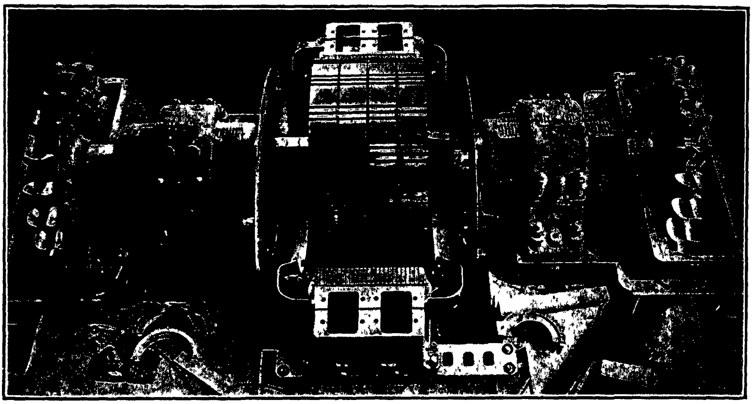


FIG. 238.—Double overhung Allis-Chalmers wheels for Pacific Light and Power Co. $D = 94''$; $h = 1{,}860'$; $N = 375$; hp. = 20,000 (for unit). (*Courtesy of Allis-Chalmers Mfg. Co.*)

two or more nozzles may be employed to give an increase in power, though at a slight sacrifice in efficiency.

The largest size impulse wheel so far as physical dimensions are concerned is an Allis-Chalmers unit with the 14-in. jet cited above. The wheel is 176 in. in diameter, develops 32,200 hp., and runs at 143 r.p.m. under a head of 800 ft.

The most powerful impulse wheel is that of the Southern California Edison Company in their Big Creek No. 2A plant under a head of 2,200 ft. This is a double overhung Allis-Chalmers unit which develops 56,000 hp. at 250 r.p.m. normally but on test developed up to 65,000 hp. One wheel of this unit is shown in Fig. 231.

The maximum efficiency attainable by a Pelton-type impulse wheel is about 88 per cent.

CHAPTER XV

THEORY OF THE IMPULSE WHEEL

179. Action of the Water.—The impulse wheel is more accurately described as a tangential water wheel from the fact that the center line of the jet is tangent to the path described by the center of the buckets. The latter is called the "impulse circle," and computations are based upon the linear velocity of the wheel at this radius. For an impulse wheel the nominal value of D is the diameter of this circle.

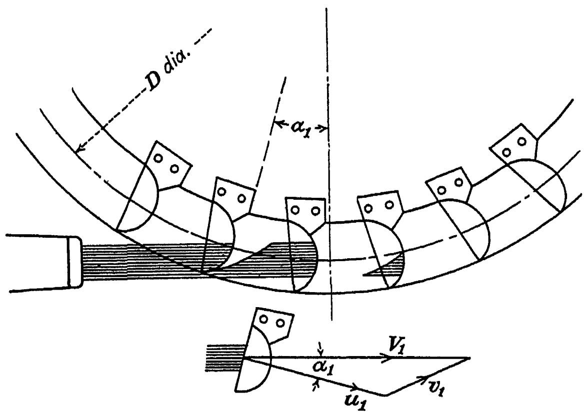


FIG. 239.

It is often stated that the jet at impact is also tangent to this circle, but this is not a true representation of the facts, as Fig. 239 will show. The jet strikes the buckets before they arrive at a point directly under the center of rotation, and hence the angle α_1 is not zero. Observation of various Pelton wheels in action has convinced the author that average values of α_1 may vary from 5 to 20 deg., according to the design of the wheel. The value of α_1 must be an average value for a given wheel from the fact that the bucket moves through a certain angle from the time when it first enters the jet until the last drop of water has struck it.

The illustration also shows that when a bucket first enters the jet it cuts off the water from the preceding bucket and leaves a "slug" of water to catch up with the latter and to complete its work upon it. Thus the water may be acting upon several buckets at the same time. This explains why there is a difference between the W' of Art. 162 and W. The former is the amount of water acting upon a single bucket; the latter is the total acting upon all the buckets. It is not necessary to know how many buckets are in action at a time, for, since the wheel does not move away from the nozzle, it follows that all of the water discharged by the nozzle may act upon it.

But it is not necessarily true that the wheel utilizes all of the water under all circumstances. Suppose, for instance, that the buckets were to move as fast as the jet; it would then be seen that none of the water could overtake them but that all would go right on through. And for speeds somewhat less than this a portion of the water would deliver its energy to the buckets, and the latter portions of the intercepted "slugs" would not be able to overtake the buckets before they had swung up above the line of action of the jet. The problem is so to design the buckets and the wheel that all of the water in the jet will be able to do its work upon the wheel when running at the proper speed. For speeds much above the normal a certain amount of water must necessarily go right on through without having had a chance to do work.

By the proper or normal speed is meant the one that the wheel should have for the jet velocity in question. A high jet velocity would require a high wheel speed, and *vice versa*. In fact, the chief concern is with the relation between the various velocities rather than with their actual values, and hence it is desirable to introduce factors that will express this relationship and be independent of the head. Thus if the jet velocity be denoted by V_1, and the linear velocity of the bucket at the impulse circle by u_1, c_v and ϕ may be used, so that

$$V_1 = c_v\sqrt{2gh}. \tag{243}$$

$$u_1 = \phi\sqrt{2gh}. \tag{244}$$

It may be seen that c_v is the velocity coefficient of the nozzle, the value of which is constant for any setting of the needle.

Thus for any given value of ϕ the relation between V_1 and u_1 is known at once regardless of the value of h.

The absolute path of the water and the velocity vectors at discharge from the buckets may be seen in Fig. 240. For different wheel speeds under the same head, which means different values of ϕ, there should be such diagrams as are shown in Fig. 241. As the speed of the wheel increases from zero, the angle of deflection of the jet continually decreases. It may also be seen from the diagrams that the value of V_2 is relatively high when the wheel is at rest, that it becomes a minimum at such a speed that α_2 is approximately equal to 90 deg., and that it then increases again.

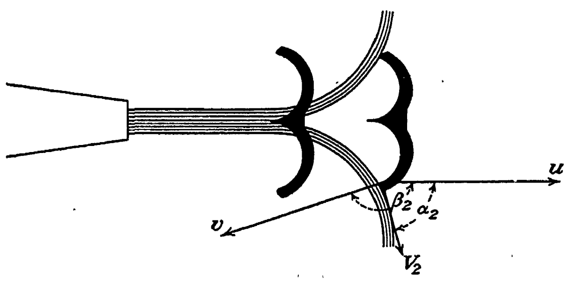


FIG. 240.

The action of the water as just described is illustrated by some rather unusual photographs taken of a 42-in. wheel in action. The side of the casing was removed for the purpose. The needle was withdrawn as far as possible, so that the maximum-size jet that the design permitted is shown in the photographs. In Fig. 241, when $\phi = 0$, the wheel was prevented from rotating by applying a sufficient torque to the shaft. The jet cannot be seen, but the water leaving the bucket is shown. In Fig. 241, when $\phi = 0.45$, the wheel is running at its most efficient speed. The water leaving the buckets drops down into the tailrace with most of its energy abstracted. In Fig. 241, when $\varphi = 0.80$, the wheel is shown at runaway speed, all load having been removed save its own friction and windage and that of the generator to which it is direct connected.

180. Force Exerted by Jet.—The tangential water wheel, Pelton wheel, or impulse wheel, as it is variously called, is really an impulse turbine with approximately "axial" flow. By this is

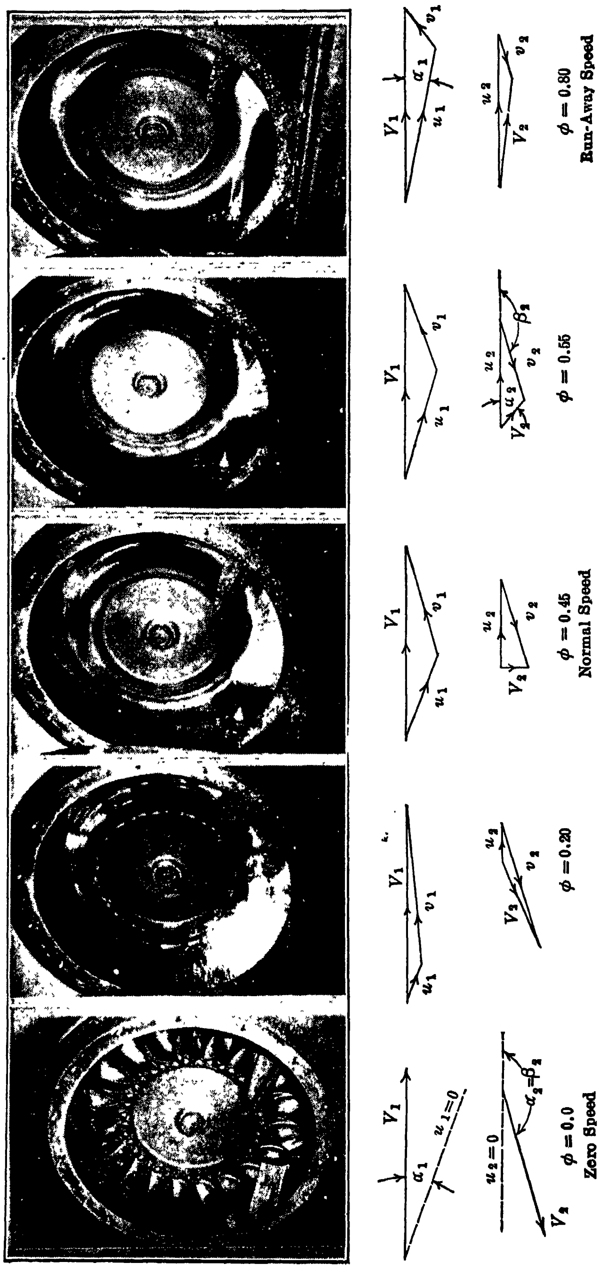


Fig. 241.—42-in. impulse wheel at different speeds under same head.

meant that $r_1 = r_2$. The latter is not strictly true but is sufficiently close for all practical purposes.

The force desired is really the tangential component of the resultant force. This may be obtained by computing the tangential component of the ΔV in Art. 157, or, since r_1 and r_2 are equal, it may be obtained from Art. 164. The results are identical. The desired component is

$$F = \frac{W}{g}(V_1 \cos \alpha_1 - V_2 \cos \alpha_2). \tag{245}$$

The values of V_2 and α_2 depend upon the jet velocity and the speed of the wheel and are therefore variable and unknown. It is desired to replace them in terms of V_1 and u_1 and wheel dimensions which may be supposed to be known. It is thus necessary to find some relation between the velocities at entrance and those at discharge. The equation of flow between these two points will be found in Art. 169, and for the impulse turbine it becomes

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} + \frac{u_2^2}{2g} - \frac{u_1^2}{2g} = k\frac{v_2^2}{2g},$$

where k is a coefficient of loss in flow over the buckets, such that the head lost is $kv_2^2/2g$. Since it is assumed that $u_1 = u_2$, then, for the Pelton wheel is obtained the special relation

$$v_2 = \frac{v_1}{\sqrt{1 + k}}.$$

In a numerical case the value of v_1 can be computed by trigonometry according to whichever one of the methods given on page 449 is deemed to be more convenient or more accurate. Having v_1, the value of v_2 may be found by the equation just given. The velocity diagram at outflow is now determined, since v_2, u_2, and β_2 are known. Thus $V_2 \cos \alpha_2$ may be found, and the value of F computed by Eq. (245). The procedure is identical with that illustrated in Arts. 162 and 164.

The numerical computation of the value of F is greatly simplified, if it is assumed that $\alpha_1 = 0$ deg., but the result is merely a close approximation to the truth.

A study of Fig. 241 shows that the value of ΔV decreases as the wheel speed increases. Assuming that $\alpha_1 = 0$ deg., it may

be shown that ΔV is proportional to $V_1 - u_1$, and thus

$$F = B(V_1 - u_1),$$

where B represents all the other factors. This is apparently the equation of a straight line between F and u_1 for any constant jet velocity V_1. Actually it is not a straight line because B is not a constant. This is due to the fact that not only is α_1 not equal to 0 deg., but it varies with the wheel speed; the factor k is not a constant for all speeds; and, above all, the amount of water

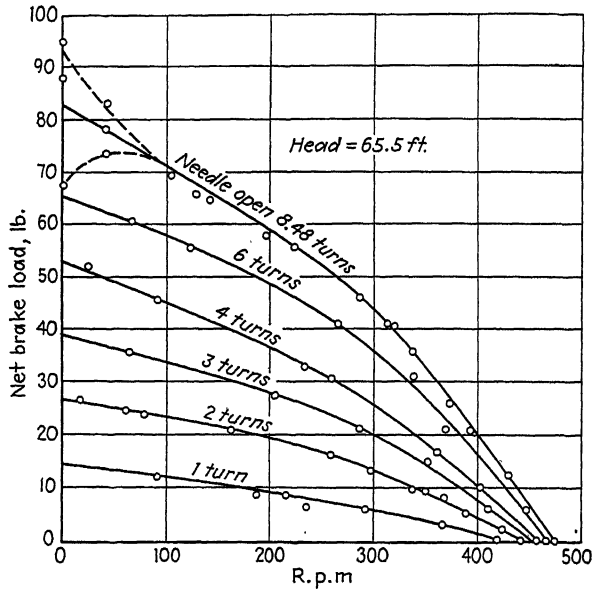


FIG. 242.—Relation between torque and speed.[1]

acting upon the wheel decreases as the speed increases above normal, as has previously been explained.

The torque exerted by the water upon the wheel may be obtained by multiplying F by r_1, the radius of the impulse circle. The torque that the wheel can deliver is somewhat less than this because of bearing friction and windage.

Figure 242 shows the performance of a certain wheel at different speeds under a constant head. The variation in the force at zero speed is due to changes in the angle α_1 and r_2/r_1 with differ-

[1] From the test of a 24-in. wheel by F. G. Switzer and the author.

ent positions of the buckets. This is shown for one nozzle setting only, though it exists in all.

181. Power of Wheel.—Since power is the product of F and u_1 or T and ω, it may be seen that it is zero when the wheel is at rest, though the torque is then a maximum, and it is also zero when the speed is a maximum, for the torque is then zero. The

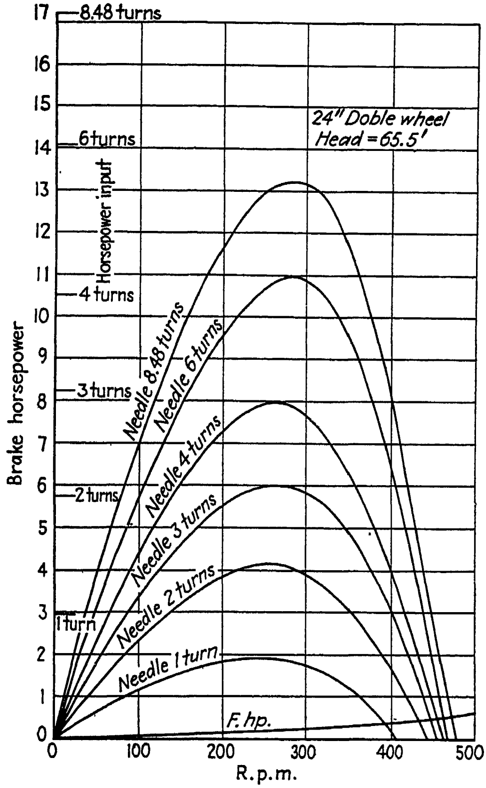


FIG. 243.—Relation between power and speed.

maximum power will be obtained for some speed between these two extremes, as shown by Fig. 243.

Since for a given head and nozzle opening the power input is constant regardless of the speed of the wheel, it follows that the efficiency is directly proportional to the power developed. But it should be noted that the power delivered in the water increases with the nozzle opening so that the needle setting that gives the largest power is not necessarily the most efficient.

182. Speed.—It may be well to emphasize that in the present discussion the word speed is being used in a relative rather than an absolute sense. Any observation on the effect of a change in the speed of the wheel u_1 is based on the assumption that the jet velocity V_1 remains the same. And by a high peripheral speed, for instance, is meant merely high as compared with that of the jet. Since, then, the principal concern is with ratios of speeds rather than with their absolute values, the factor ϕ could better be used, as defined in Eq. (244).

Using the expression $F = B(V_1 - u_1)$, it should be concluded that F would become zero when $u_1 = V_1$, or when $\phi = c_v$, the value of which would be about 0.98. Also, if u_1 should be used as multiplier, the value of the power would be

$$P = Fu_1 = B(V_1 - u_1)u_1;$$

and from this the power would appear to be a maximum when $u = 0.5V_1$. But, as has been stated, the factor B is not a constant. Because of the large amount of water that is not effective upon the wheel at high values of ϕ, and also because the bearing friction and windage prevent the torque from ever being reduced to zero, the actual maximum speed attained by the tangential water wheel is such that the maximum value of ϕ is approximately 0.80.

In like manner the maximum power, and hence the maximum efficiency for a given nozzle opening, is also attained when the wheel speed is something less than $0.5V_1$. Thus in actual practice for the best efficiency

$$\phi_e = 0.43 \text{ to } 0.48.$$

In practical applications the performance of a wheel at a constant speed under a constant head is usually of most interest. Values for this may be obtained from Figs. 242 and 243 by following along any vertical line. Generally, the vertical line should be the one for the speed at which the maximum efficiency is found. The resulting curves for the impulse wheel would be very similar to those for the reaction turbine shown in Fig. 274.

183. Head on Impulse Wheel.—The nozzle is considered an integral part of the impulse wheel, and hence the head under which the wheel is said to operate must include it. If C in Fig. 244 indicates a point at the base of the nozzle,

$$h = H_c = \frac{p_c}{w} + \frac{V_c^2}{2g}. \tag{246}$$

It is this value of h that should be used in determining the efficiency of the wheel.

This value of h is the total fall from headwater to nozzle minus the head lost in the pipe line. The energy supplied at this point is expended in four ways. A small amount is lost in flow through the nozzle, a portion is expended in hydraulic friction and eddy losses within the buckets, kinetic energy is carried away in the water discharged from the buckets, while the rest is delivered to

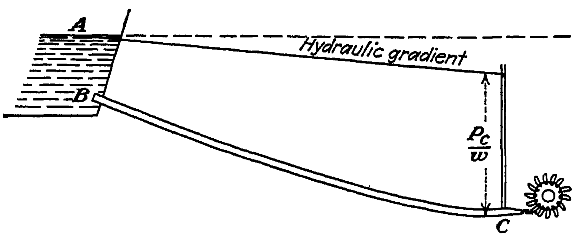


FIG. 244.

the wheel to do useful work and overcome mechanical friction and windage losses. Calling h'' the head delivered to the buckets,

$$h = \left(\frac{1}{c_v^2} - 1\right)\frac{V_1^2}{2g} + k\frac{v_2^2}{2g} + \frac{V_2^2}{2g} + h''. \tag{247}$$

The head utilized h'' may be determined as shown in Art. 166. The procedure is similar to that for the computation of a value of F.

184. PROBLEMS

339. A nozzle having a velocity coefficient of 0.98 discharges a jet 6 in. in diameter under a head of 800 ft. This jet acts upon a wheel with the following dimensions: diameter, 6 ft.; $\alpha_1 = 10$ deg.; $\beta_2 = 165$ deg.; and it is assumed that $k = 0.70$. Find the force exerted upon the buckets when $\phi = 0.45$. *Ans.* $F = 17{,}600$ lb.

340. Solve the foregoing, assuming that $\alpha_1 = 0$ deg., and find the power developed upon the buckets of the wheel. What is the hydraulic efficiency of the wheel? If the mechanical efficiency of the wheel is 0.97, what is the gross efficiency? *Ans.* $F = 17{,}750$ lb., 3,290 hp., 0.833, 0.807.

341. Assuming $\alpha_1 = 0$ deg. in Prob. 339, find the power lost in hydraulic friction within the buckets. Find the value of V_2 and determine the power carried away in the water discharged from the turbine.

Ans. 460 hp., 57.5 hp.

342. A good proportion between jet and wheel is that the diameter of the wheel in feet should equal the diameter of the jet in inches. Using this ratio, what size wheel would be required to deliver 5,000 hp. under a head of 1,400 ft., assuming an efficiency of 82 per cent? What would be the speed of the wheel? *Ans.* 4.88 ft., 563 r.p.m.

343. A Pelton wheel has the following dimensions: $\alpha_1 = 15$ deg., $\beta_2 = 170$ deg., $r_1 = r_2$, $k = 0.5$. If the jet diameter equals 5 in., and the jet velocity is 300 ft. per sec., find the power output when $u_1 = 0.45\ V_1$ if the mechanical efficiency is 0.96. *Ans.* 5,475 b.hp.

344. A Pelton wheel 7 ft. in diameter runs at 360 r.p.m. under a head of 1,300 ft. The velocity coefficient of the nozzle is 0.98, and the jet diameter is 6 in. The bucket angle $\beta_2 = 160$ deg., and it is assumed that $\alpha_1 = 0$ deg. and $k = 0.80$. Find the power in the jet, the power delivered to the wheel, the power lost in hydraulic friction, and the power lost in discharge from the wheel. *Ans.* 7,880, 6,670, 1,000, 216 hp.

345. Prove that the exact equation for all impulse turbines is

$$F = \frac{W}{g}\left[V_1 \cos \alpha_1 - x^2 u_1 - \frac{x \cos \beta_2}{\sqrt{1+k}}\sqrt{V_1^2 + x^2 u_1^2 - 2V_1 u_1 \cos \alpha_1}\right],$$

where $x = r_2/r_1$ and $F(= T/r_1)$ is an equivalent concentrated force. For the Pelton wheel or axial-flow turbine $x = 1$; for the outward- or inward-flow Girard impulse turbines it is more or less than unity, respectively.

346. Prove that for a Pelton wheel with $r_1 = r_2$ and assuming $\alpha_1 = 0$ deg. the expression for force exerted is

$$F = \left(\frac{W}{g}\right)\left(1 - \frac{\cos \beta_2}{\sqrt{1+k}}\right)(V_1 - u).$$

347. Prove that for the Pelton wheel, with restrictions as in the preceding problem, the hydraulic efficiency is given by

$$e_h = 2\left(1 - \frac{\cos \beta_2}{\sqrt{1+k}}\right)(c_v\phi - \phi^2)$$

and is thus independent of the head.

348. The maximum speed attained by the wheel of Fig. 242 was 475 r.p.m. under a head of 65.5 ft. What was the value of ϕ?

The best efficiency for the wheel whose curves are shown in Fig. 243 was found with the needle open six turns. The speed was 275 r.p.m., and the head 65.5 ft. What was the value of ϕ_e?

CHAPTER XVI

DESCRIPTION OF THE REACTION TURBINE

185. The Reaction Turbine.—With the impulse turbine, in general, water may be admitted by a series of nozzles around the

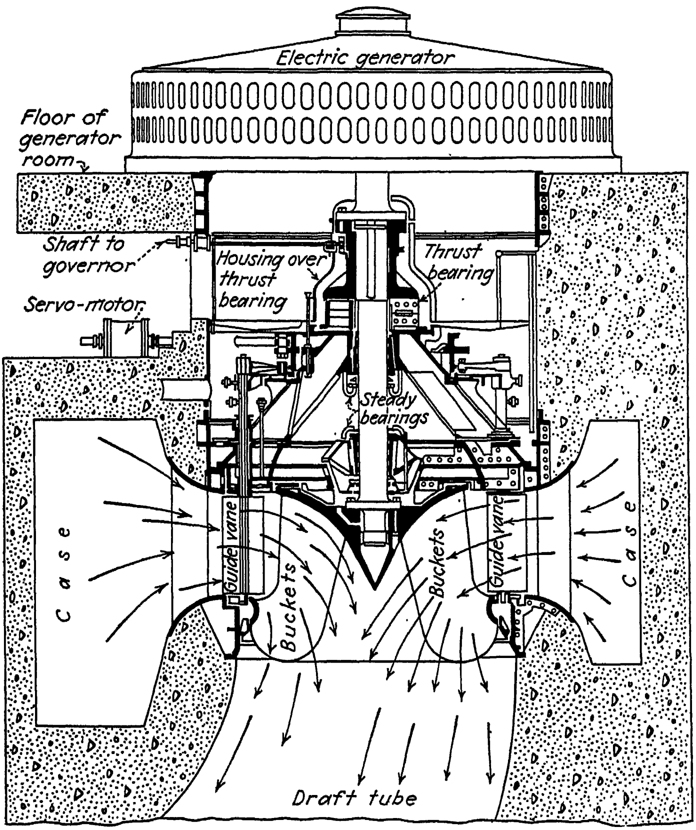


FIG. 245.—Reaction turbine.

circumference of the wheel, but with the usual type of Pelton wheel one nozzle only is employed, though occasionally two or more may be used. Thus only a small portion of the buckets

on the wheel are in service at a time. But by definition the passages of a reaction turbine must be completely filled with water, and in order that this condition may be fulfilled it is necessary that water be admitted around the entire circumference. Since every vane or bucket is in continuous use, it may be seen that

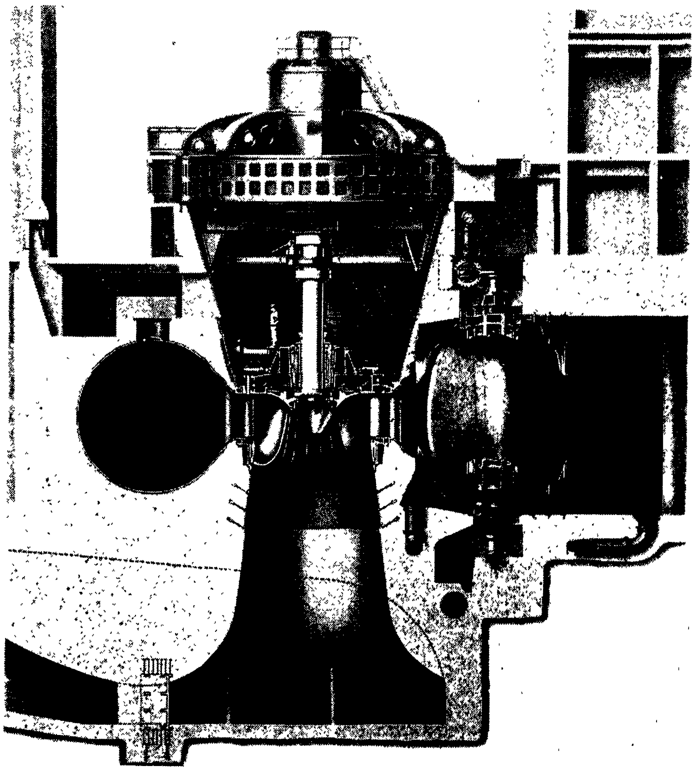


FIG. 246.—54,000-hp. turbine for Conowingo plant of Susquehanna Power Company under head of 89 ft. $N = 81.8$ r.p.m. (*Courtesy of Allis-Chalmers Mfg. Co.*)

more power may be developed upon a reaction turbine of a given diameter than is the case with a Pelton wheel.

The general arrangement of one type of reaction turbine is shown in Fig. 245. Water under pressure enters a spiral casing which encircles the unit. Since water is flowing into the runner around its entire circumference, the cross-sectional area of this casing may decrease in proportion to the decreasing volume of water to be handled. From the casing the water flows through

passageways between guide vanes which serve to give it a direction as near tangential as is practicable, and then it flows through the runner and into the draft tube. A cross section of the runner showing the buckets or vanes may be seen in the figure. It may be noted, however, that the vane, being a warped surface, will not lie in the plane of the paper and hence must be portrayed by circular projection. That is, points along both edges may be rotated about the axis until they fall in the plane of the paper.

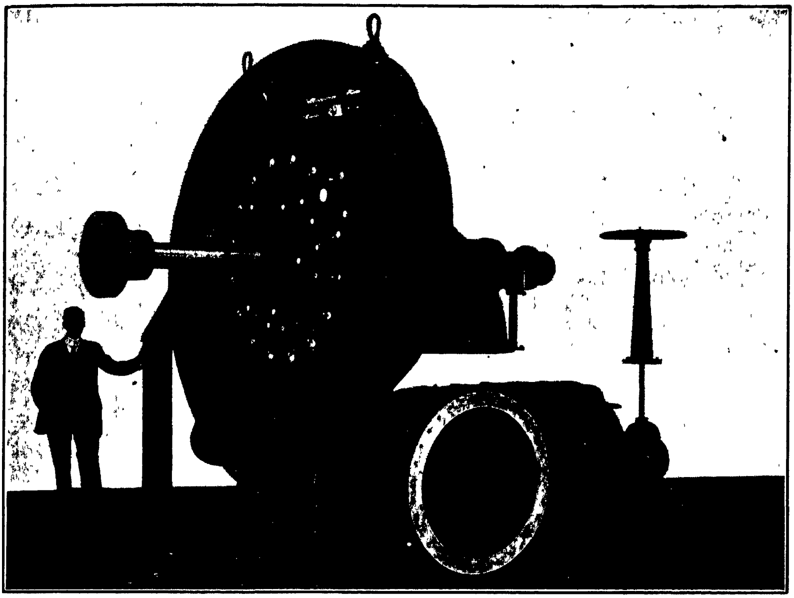


FIG. 247.—Spiral-case turbine showing main gate valve shifting ring and links for guide vanes, and draft elbow.

In most cases, the entrance edge may lie in one plane containing the axis of rotation, and the discharge edge may lie in another plane also containing the axis, but this is not always true. The view of the vane or bucket shown is thus not a true cross section and is more properly termed a *profile.*

Additional features shown in Fig. 245 are the thrust bearing which carries the entire rotating weight of turbine and generator, the servo motor, and the stay vanes. The servo motor is a cylinder containing a piston actuated by oil under pressure and controlled by the governor. It serves to operate a mechanism which rotates each guide vane about its own axis for the purpose

of regulating the water supplied to the wheel. In the case of a large unit, such as shown in this figure, the stay vanes are introduced as columns to assist in sustaining the load above. But instead of being round columns, which would produce eddy currents, they are made as vanes with a curvature conforming to the natural streamlines. These stay vanes are outside the guide vanes, and the assembly, surrounding the guide vanes, is sometimes called a *speed ring*, though for no apparent reason.

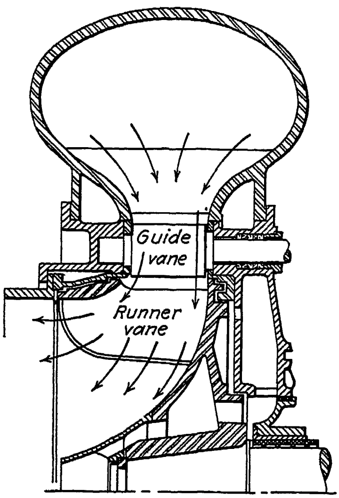


FIG. 248.—Labyrinth seal.

A view of another very large turbine is shown in Fig. 246.

The shaft need not necessarily be vertical but may be horizontal, as in Fig. 247. The latter also shows a casing of metal instead of one formed in concrete, but the relative arrangements of case, guides, and runner and the conditions of the flow of water through it are the same whatever the setting. The difference is purely mechanical.

The leakage of water from the casing to the draft tube past the runner cannot be prevented but may be kept within reasonable limits by having a small clearance between the rotating runner and the case. The construction in Fig. 248 minimizes this leakage by introducing a tortuous labyrinth passage for the water to traverse.

The arrangement of another reaction turbine with a horizontal shaft may be seen in Fig. 249. This particular one is of the open-flume type and is set so as to be completely surrounded by water in a manner similar to the vertical-shaft turbine shown in Fig. 266. The water flows through the stationary guide vanes and enters the runner, which is in the center. In Fig. 249 there are two runners set on the same shaft and discharging into a common draft chest, from which the water flows down to the tailrace through a draft tube.

186. Runners.—The part of the turbine upon which the water does its work is called the runner. Runners may be built up of separate pieces of metal which are welded together, but they

are usually cast in one piece. Occasionally, they are built in sections, and the sections bolted together. For large sizes and

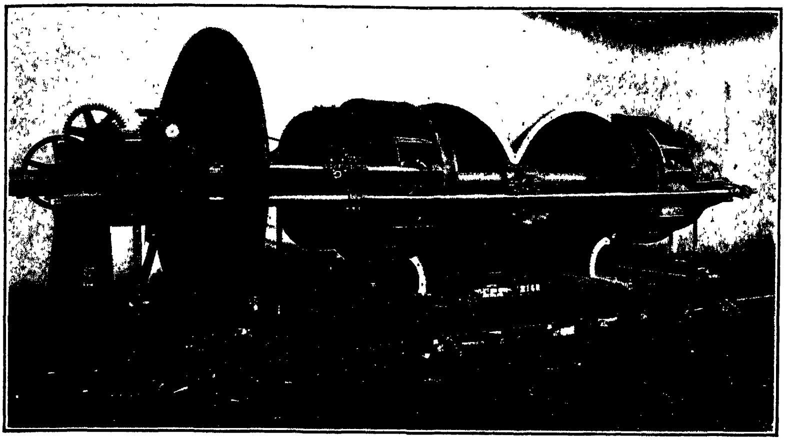


FIG. 249.—Turbine with cylinder gates for open flume. (*Courtesy of Platt Iron Wks. Co.*)

FIG. 250.—Low-speed turbine runner. $D = 74''$; $h = 487'$; $N = 360$; hp. = 20,000. (*Courtesy of Pelton Water Wheel Co.*)

low heads, cast iron is employed. Better runners are made of bronze, and occasionally cast steel is used for high heads.

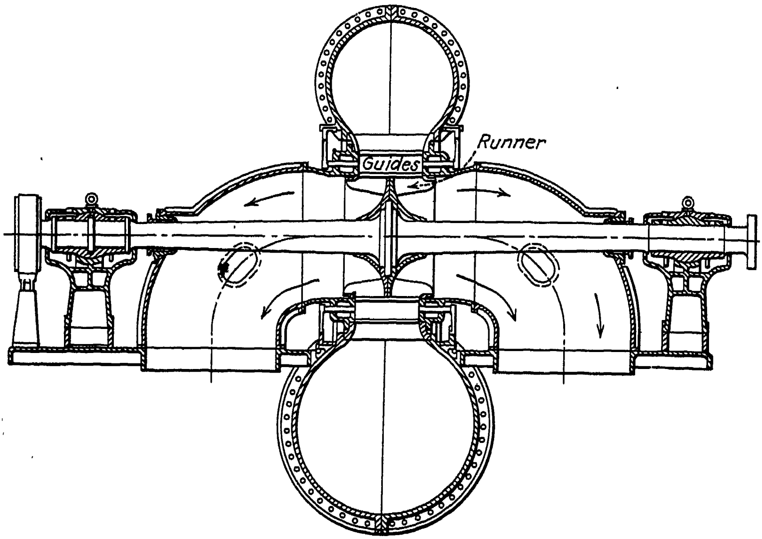


FIG. 251.—Double-discharge turbine

FIG. 252.—One of the world's largest turbine runners. For Cedar Rapids Mfg. and Power Co. $D = 143''$; $h = 30'$; $N = 55.6$; hp. = 10,800.

Runners differ considerably in their proportions and appearance. One extreme is shown in Fig. 250, while the other is shown in Figs. 252, 253, and 254. It may be noted that the runners in Figs. 250 and 253 develop the same amount of power though differing widely in size. This is due to the fact that the smaller runner operates under a much higher head and consequently

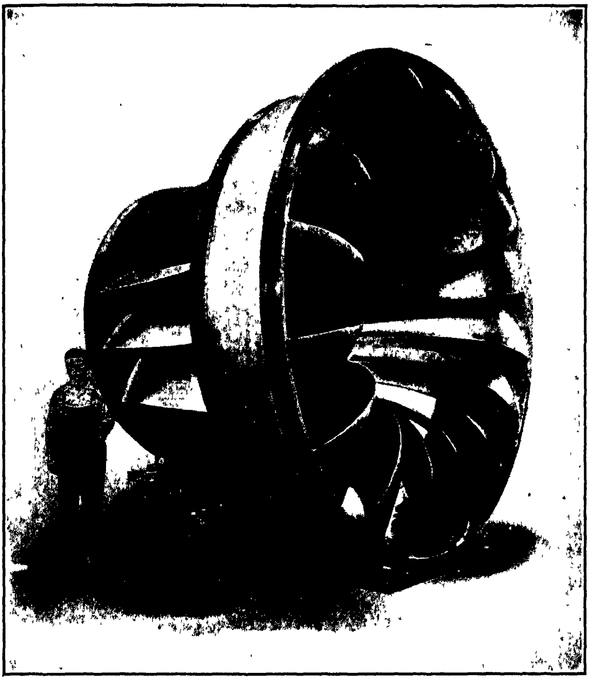


FIG. 253.—High-speed turbine runner. $D = 102''$; $h = 76'$; $N = 120$; hp. = 20,000. (*Courtesy of I. P. Morris Co.*)

needs to discharge less water for the same amount of power. And a still larger runner, which is shown in Fig. 252, develops less power than either of the others because it is under a still lower head.

It may be noted that the width of the runner parallel to the shaft in Fig. 250 is a very much smaller proportion of the diameter of the runner than in the type shown in Fig. 252.

Sometimes runners are of the double-discharge type, as in Fig. 251, which is equivalent to placing two single discharge runners back to back. Such a turbine must have two separate draft elbows.

The nominal diameter D of a turbine runner is that shown in Fig. 256 (*a*) and (*b*) and is not necessarily the maximum value. This is the dimension that will be found in the title of Fig. 252, for instance, but the maximum diameter is 17 ft. 7 in.

Since the runner always moves in a direction opposite to that of the relative velocity at discharge, it may be seen that the runners in Figs. 250 and 253 would turn counterclockwise, as viewed in the picture, while that shown in Fig. 252 would turn clockwise if viewed from above, as does likewise the one in Fig. 254.

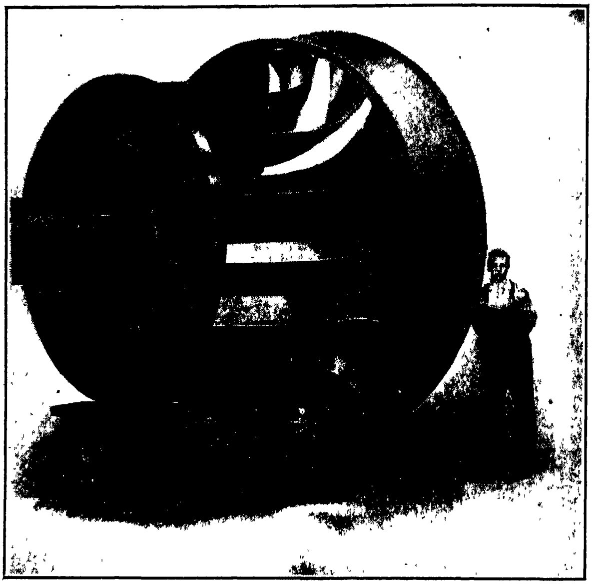


FIG. 254.—Turbine runner for Laurentide Co.

The first reaction turbines were of the outward-flow type; that is, the water enters the runner on the inside and flows toward the outer radius. But the inward-flow type is the only one that is of any importance at the present time, all others having been eliminated because of certain disadvantages, mostly of mechanical construction.

An inward-flow turbine runner was proposed by Poncelot in 1826, but the first one was built and patented by Howd in 1838. In 1849 J. B. Francis constructed a pair of pure radial inward-flow turbines from the Howd patent, but his wheels were of much better design and workmanship. Because of the publicity given

to these wheels, due to the very precise tests that he conducted on them, his name became attached to them, and today modern inward-flow turbines are known as Francis turbines, even though they may differ considerably from his original design, which is shown in Fig. 255 and was of a pure radial-flow type.

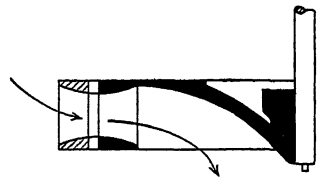


FIG. 255.—Original Francis turbine.

By radial flow is meant that a particle of water, during its flow through the rotating runner, remains in a plane normal to the axis of rotation, so that its position changes only with respect to its distance from the axis of rotation. In the evolution of the modern turbine it became desirable to have the water enter the runner with a "radial" flow and then turn and flow in such a manner that a component of its velocity might be parallel to the shaft. In fact, some of the particles of water, at least before

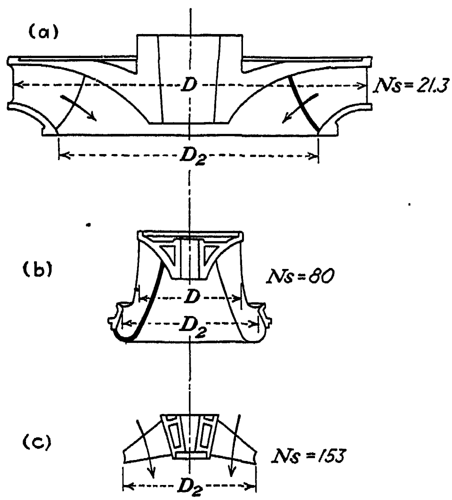


FIG. 256.—Relative sizes of different types of runners for same power under same head.

they reached the discharge edge of the bucket or vane, might be following paths that lay on the surfaces of cylinders concentric with the axis. This is known as mixed flow, and such a type of turbine is sometimes called the American turbine, though the

name Francis is generally extended to cover all inward-flow wheels. In Figs. 250 and 256 (*a*) may be seen the nearest approach in present practice to a radial-flow runner, while in Figs. 253 and 256 (*b*) may be seen the mixed-flow type.

The evolution of the type in Fig. 256 (*b*) was due to the demand for both higher power and speed. For a given head a higher

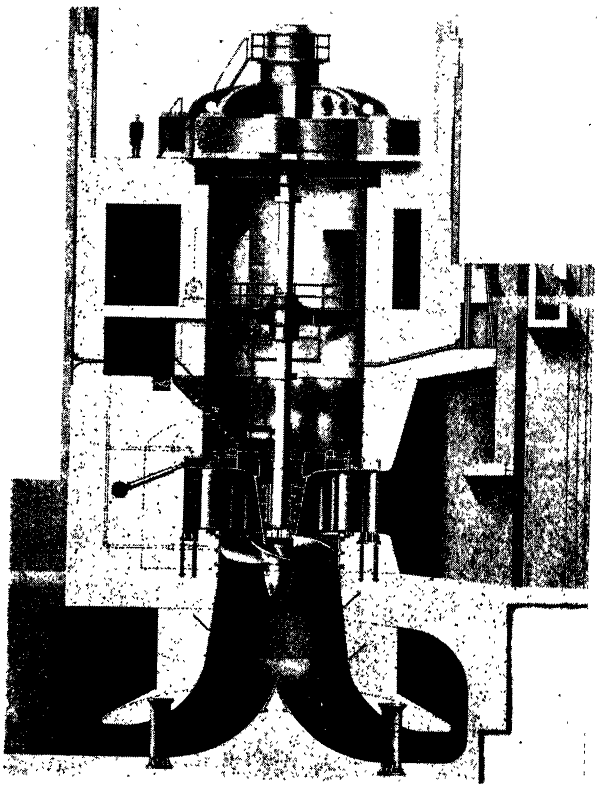


Fig. 257.—Propeller turbine. (*Courtesy of Allis-Chalmers Mfg. Co.*)

power requires a larger diameter, but a higher rotative speed requires a smaller diameter. In order to meet these conflicting requirements, it was therefore necessary to modify the type shown in Fig. 256 (*a*) by decreasing the diameter to secure higher rotative speed and at the same time extending the dimension parallel to the axis in order to maintain the capacity. Certain other alterations which do not appear in this profile view accompanied these changes, so that the same power under the same head may

be obtained with runners that differ from each other very materially in appearance and in regard to their proper rotative speed. With a continued demand for still higher power and speed under low heads, a new type indicated in Fig. 256 (*c*) has now been produced, called the propeller type, because it resembles, to some extent, a ship propeller. The flow through it is practically axial;

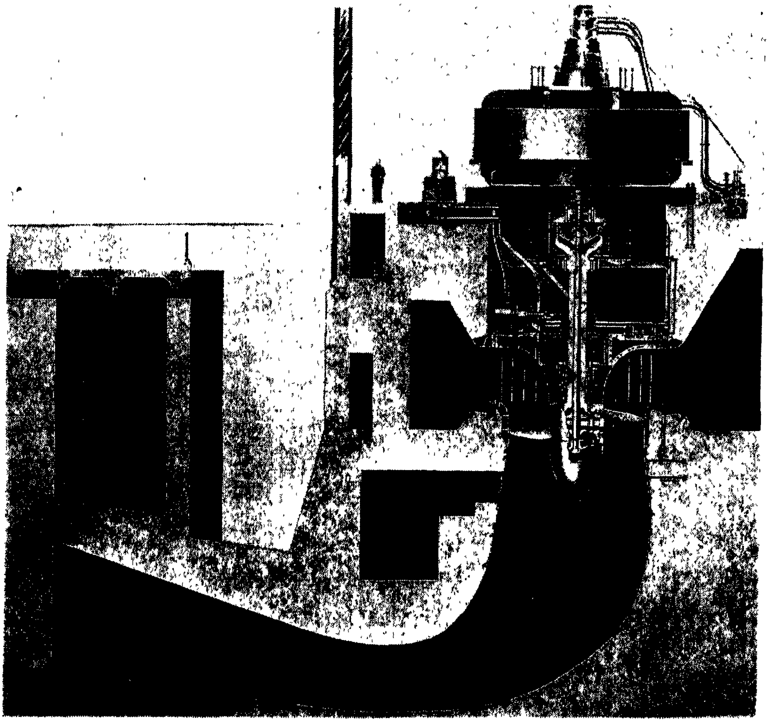


FIG. 258.—Propeller turbine for Safe Harbor Water Power Corporation. (*Courtesy of I. P. Morris division of Baldwin-Southwark Co.*)

that is, each particle of water moves in a path which is at approximately a constant radius from the axis of rotation of the wheel. Propeller-type turbines are shown in Figs. 257 and 258.

In Fig. 256 each runner will develop the same power under the same head, but the size will vary as shown. The actual rotative speed is proportional to the numerical values there designated by N_s. The latter is not, however, the actual speed of the runner but rather a factor indicating the type of the turbine. Although

called *specific speed,* it involves the head and power as well as the speed of rotation, as may be seen in Art. 201.

The runner in Fig. 250 is described as a low-speed runner, though its actual speed is 360 r.p.m.; while that in Fig. 252 may be called a high-speed runner, though it runs at only 56.6 r.p.m. But the use of these terms is relative. Thus if a turbine of the

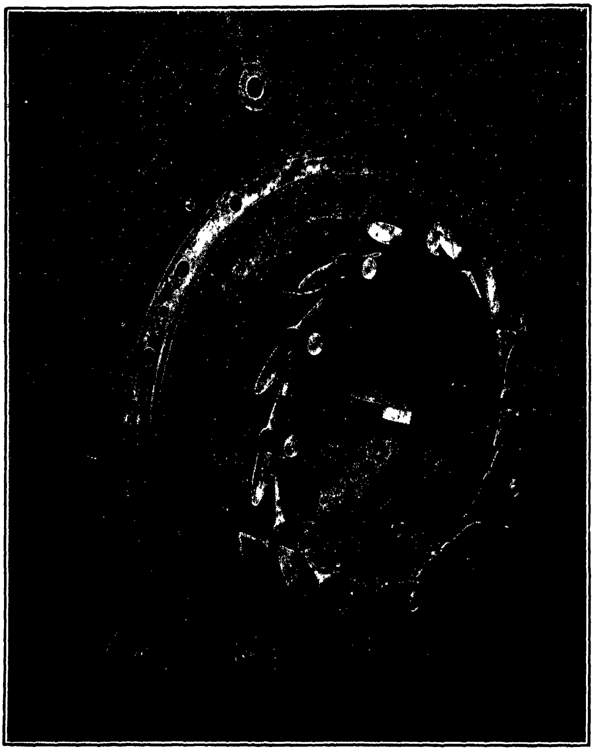


Fig. 259.—Wicket gates or swing gates. (*Courtesy of Pelton Water Wheel Co.*)

first type were to be built to deliver the same power under the same head as the second, it would run at only 17 r.p.m. Hence our terms of *low speed* for the former and *high speed* for the latter are seen to be justified. If it were physically possible to use a Pelton wheel for the same power and head as the turbine in Fig. 252, it would run at only 3 or 4 r.p.m. Thus the Pelton wheel may be seen to be a lower speed wheel even than any reaction turbine. If actual rotative speeds encountered in practice are higher, it is because such wheels are used under higher heads.

187. Gates and Governing.—The quantity of water passed through the turbine is regulated by means of gates, of which there are several kinds. In Fig. 249 the cylinder gate is used. In that class of turbine the guide vanes surrounding the runner are absolutely fixed. Between the ends of these vanes and the runner is a metal cylinder which may slide along parallel to the shaft. If moved in one direction, it admits water to the runner and may be so far withdrawn as to offer no obstruction whatever between the guides and the wheel. And if it is moved in the

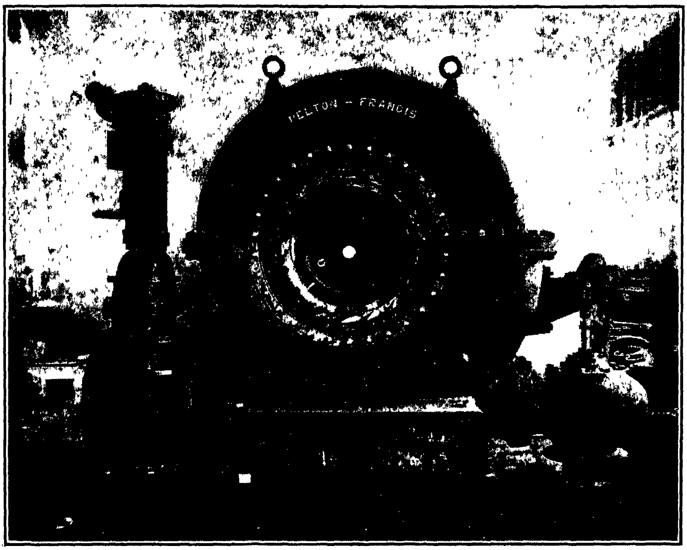


FIG. 260.—Spiral-case turbine showing swing gates. (*Courtesy of Pelton Water Wheel Co.*)

other direction, it is possible to shut off the water altogether. This style of regulation causes the turbine to have a poor efficiency on *part gate,* the term used when the turbine is running under less than full load. But such a style of gate permits a turbine to be constructed at less cost.

The better type of gate so far as efficiency is concerned is the kind shown in Fig. 259. Here the guide vanes themselves are movable, and by rotating about their axes they may vary the size of the area through which water may flow. This means that the angle α_1 changes. These gates are known as swing gates, wicket gates, or pivoted guide vanes. They involve more expen-

sive construction than the cylinder gate but are vastly better if economy of water is any object.

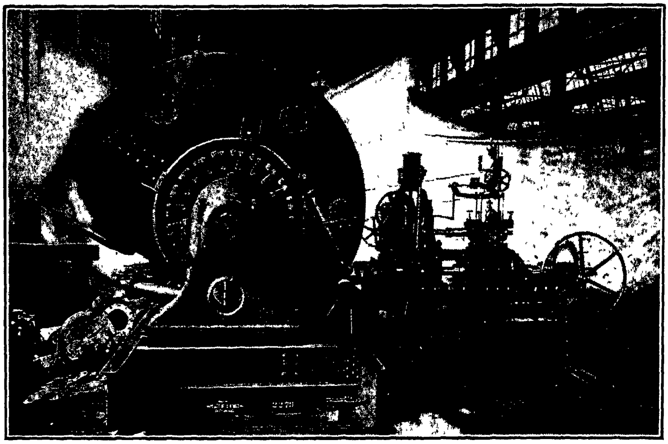


FIG. 261.—Shifting ring for operating gates. (*Courtesy of Platt Iron Wks. Co.*)

In Fig. 260 may also be seen some movable gates as they are installed in the turbine. The runner is to go into the space in the center.

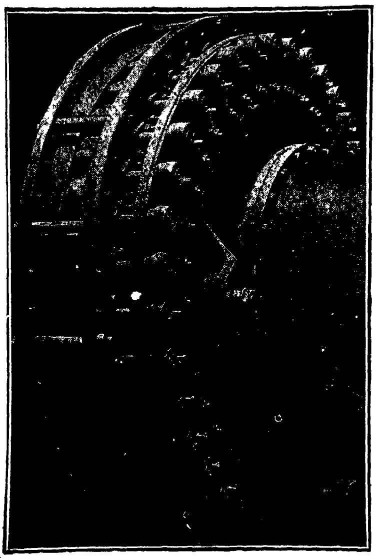


FIG. 262.—Swing gates. (*Courtesy of Allis-Chalmers Mfg. Co.*)

The swing gates are operated by moving a "shifting ring" to which each gate is attached by links. In Fig. 261 may be seen the rods from the governor connected to this ring so that, when it is moved slightly with the turbine shaft as a center of rotation, each gate will be turned through an angle. The links that connect the gates to this ring can be seen more clearly in Fig. 262.

The problem of governing a reaction turbine is similar to that of the impulse wheel. When the governor closes the gates and thus reduces the discharge through the turbine it is necessary to provide a by-pass for the water in order to prevent water

hammer in the pipe line. The usual practice is to use a relief valve such as that shown over at the right in Fig. 263. When the governor closes the gates it opens the relief valve at the same time, and the water coming down the pipe is then discharged through this into the tailrace alongside of the draft tube. The action of such relief valve may be seen in Fig. 264.

FIG. 263.—Tallulah Falls turbine showing gate mechanism and relief valve. $h = 580'$; $N = 514$; hp. $= 19{,}000$. (*Courtesy of S. Morgan Smith Co.*)

The connection between the governor and the relief valve is usually not a rigid connection, in order that the relief valve may slowly close and prevent the waste of water.

188. The Draft Tube.—The water is conducted from the turbine to the tailrace through a draft tube, which may be constructed of riveted-steel plates, as in Fig. 264, or molded in concrete, as in Fig. 267. The draft tube should be made airtight so that a partial vacuum can exist within it, and a "suction" thus produced on the discharge side of the runner which will compensate for the elevation of the latter above the tailwater

level. By the use of the draft tube it is possible to set the turbine at a convenient distance above the water level without losing any head thereby.

But this is not the sole function of the draft tube. The velocity with which the water is discharged from the runner represents kinetic energy that is not utilized, and such a loss cuts down the

FIG. 264.—Discharge from relief valve when opened.

efficiency of the wheel. If the draft tube is made to diverge, the velocity at its mouth will be much less than that with which the water enters it from the runner, and hence the kinetic energy finally lost may be much reduced. With some types of turbine runners it is necessary to allow the water to be discharged with a relatively high velocity, and such wheels would not possess favorable efficiencies if it were not for the use of suitable draft tubes. The usual rate of diffusion provided for is such that a circular tube will be made a frustum of a cone the vertex angle

of which is 8 deg. Some experiments by the author indicate that a larger angle than this might be permissible. For a given rate of diffusion the longer the tube the greater the reduction of the kinetic energy of the water. Therefore, in some cases it is desirable to have a long draft tube even though the runner might be set very near the water level.

On account of the function that is fulfilled by the draft tube it is properly regarded as an integral part of the turbine. Considering the turbine and the draft tube as a unit, it may be seen

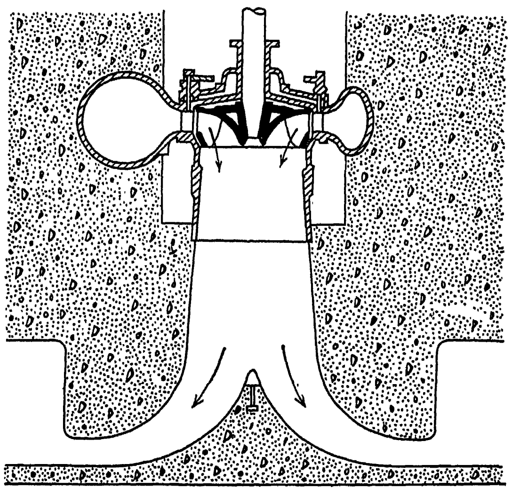


FIG. 265.—Moody spreading draft tube.

that the less the kinetic energy lost from the mouth of the tube the higher the efficiency of the wheel, thus justifying the statement of the preceding paragraph. But it is not yet clear just how this saving in the draft tube enables the turbine itself to deliver more power until the effect of the draft tube upon the pressure at the exit from the runner is considered. The less the losses within the draft tube and the less the discharge loss from the draft tube the less the pressure will be at the point of discharge from the runner, thus increasing the effective head on the runner.

With a low-speed type of turbine, the water discharging from the runner at full load may have very little, if any, rotation, but at any other gate opening it may be found that the water is rotating in the draft tube as well as flowing parallel to the axis

of the latter. And in the case of modern high-speed wheels, there is always a condition of whirl to be found in the draft tube at all loads. A condition is thus found approaching that of the free vortex described in Art. 57. In order to conserve and

Fig. 266.—Reaction turbine in open flume. (*Courtesy of S. Morgan Smith Co.*)

efficiently transform the whirl component of the velocity, the type of draft tube shown in Fig. 265 may be used. In some cases, where mechanical construction permits, it would be desirable to have this inner cone extend clear up to the runner itself. The reason for this is that, according to Art. 57, as the radius diminishes the velocity of whirl increases, and consequently the pres-

sure decreases. But it cannot fall below the vapor pressure of the water, and consequently an unstable and turbulent condition is established, which the introduction of a solid core prevents. It

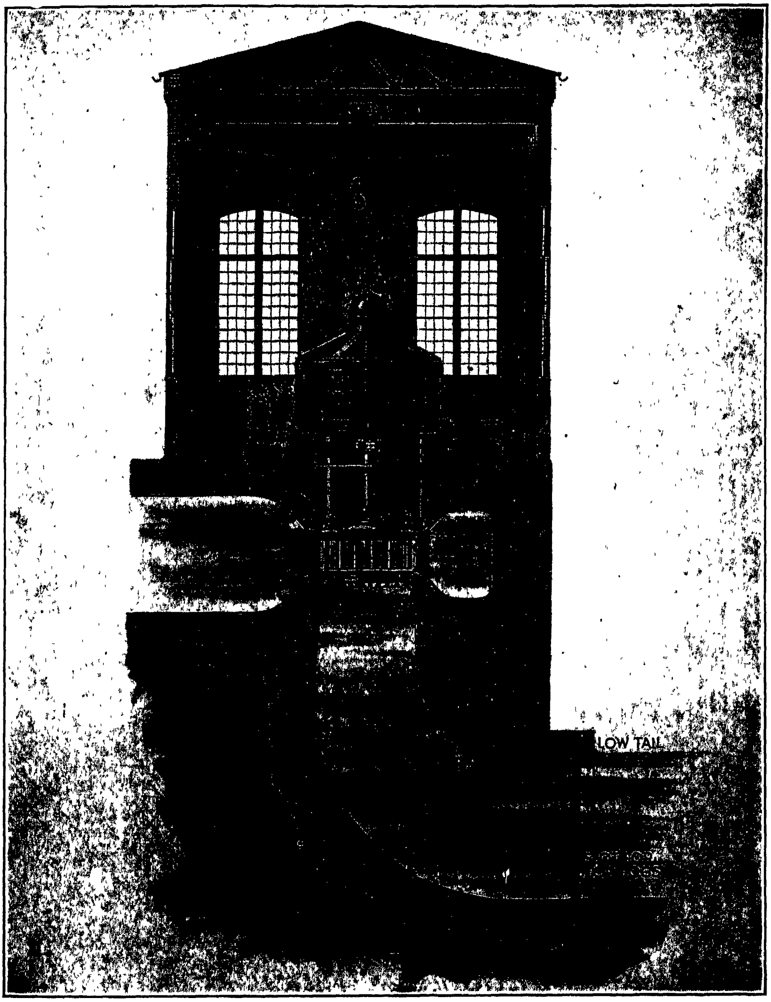


FIG. 267.—Reaction turbine in concrete case. (*Courtesy of S. Morgan Smith Co.*)

may be seen that, as the water spreads out at the bottom of the tube, the same case exists as pictured in Art. 57.

A similar effect of the Moody "spreading tube" is secured by the "hydraucone" draft tube of W. M. White where the cone is assumed to be formed of "dead" water.

189. Cases and Settings.—The turbine, draft tube, and all parts intimately connected with it comprise what is called the setting. Impulse wheels are almost always set with horizontal

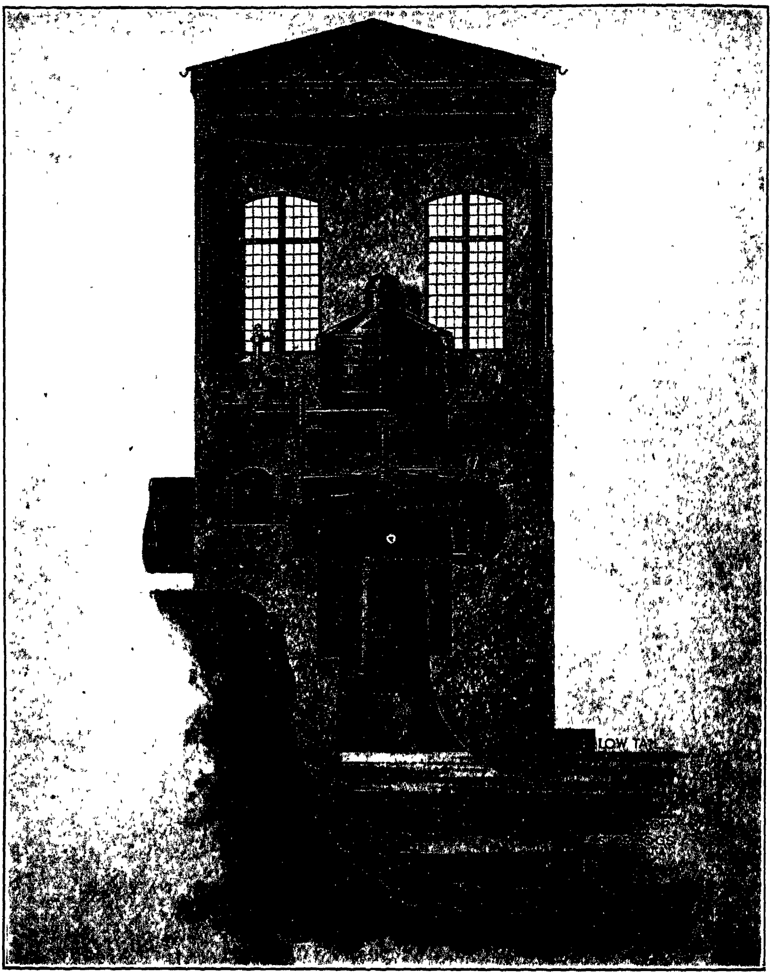


FIG. 268.—Reaction turbine in metal case. (*Courtesy of S. Morgan Smith Co.*)

shafts, but reaction turbines may have either horizontal or vertical shafts. For large units under low heads the vertical shaft is the most recent practice, as it permits several desirable features to be attained.[1] Occasionally, several runners may be mounted

[1] TAYLOR, H. B., "Present Practice in Design and Construction of Hydraulic Turbines," *Can. Soc. Civil Eng.*, Jan. 15, 1914.

on the same shaft, but the tendency is to eliminate such construction and have larger runners and fewer of them; and two on the same shaft, as in Fig. 249, is as many as are desirable. As in

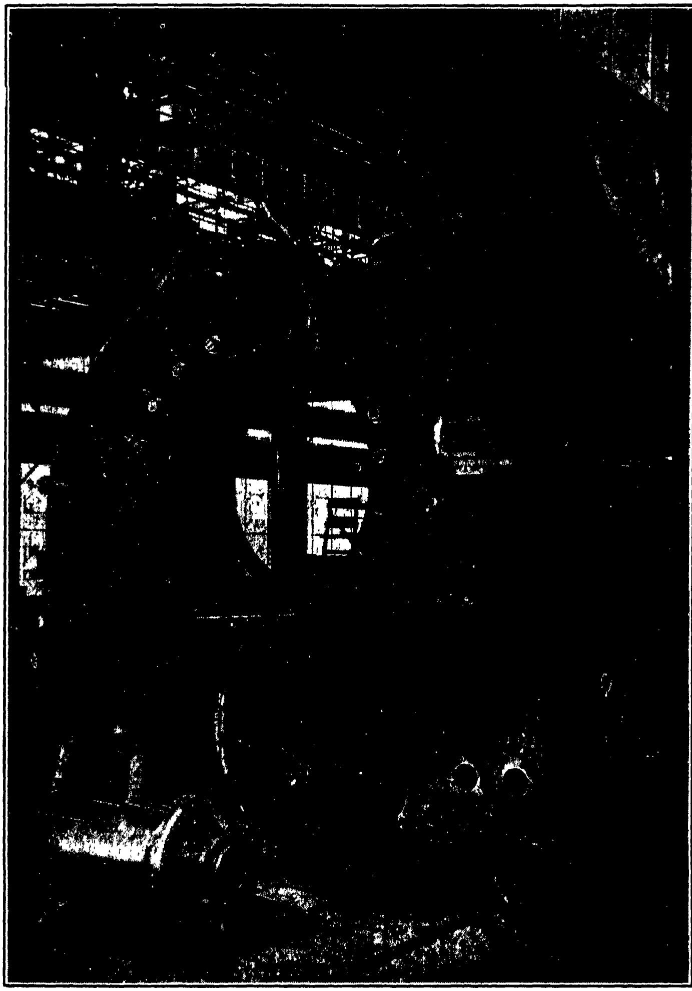


Fig. 269.—Large spiral case for Canadian Light and Power Co. in shop of I. P. Morris Co.

the case of the impulse wheel, two separate turbines may be connected to a single generator.

Under low heads of not more than 20 or 30 ft. the open-flume setting, such as is shown in Fig. 266, is sufficient, but for higher

heads this is not practicable. For either low or moderate heads the wheel may be enclosed within a concrete case, as in Fig. 267, but the action of the turbine is no different from that in the preceding case. The water has no free surface immediately above the turbine, but it is under practically the same pressure as if it did have. The only difference is that, since the area of the water passage is less than before, the velocity with which the water approaches the turbine will be somewhat higher, and thus there will be a lesser rate of acceleration as the water enters the guide vanes. For still higher heads a concrete case would be unsuit-

FIG. 270.—Intake for turbine at Keokuk. (*Courtesy of Mississippi River Power Co.*)

able, and then the guide vanes are surrounded by a metal case, as shown in Fig. 268. The only difference between this and the other two cases, so far as the hydraulics is concerned, is largely one of appearances, save that the velocity of the water as it approaches the guides may be somewhat higher owing to the smaller area.

In order that the water may have the same velocity of approach to the guides all around the circumference, the spiral case is frequently used. Cases of this type are illustrated in Figs. 247, 248, 268, and 269. In Fig. 247 may also be seen the main gate valve which may be used to shut off the water more completely than is possible with the wicket gates, and on the right-hand side may be seen a portion of the draft elbow. Very large cases

are built in sections, as shown in Fig. 269. The spiral case is considered the most desirable type, though other less expensive ones are sometimes used.

In Fig. 270 a glimpse is given into the intake of a large turbine set, as in Fig. 267. In such a setting the runner and guide vanes may be surrounded by a set of stay vanes, also called a *speed ring,* such as shown in Fig. 271. The columns which support the upper crown plate and its load are made of a shape similar

FIG. 271.—Speed ring for Canadian Light & Power Co. in shop of I. P. Morris Co.

to guide vanes so as to reduce eddy losses and also to give the water the proper direction as it enters the real guide vanes.

190. Conditions of Service.—The reaction turbine is well adapted for service under moderately low heads, especially for large powers, but the propeller type is even more suitable for extremely low heads coupled with very large amounts of power.

Reaction turbines may also be used satisfactorily for heads of several hundred feet. The highest head so far employed for a reaction turbine is 1,190 ft. for an installation in Italy. The capacity is 7,530 hp., and the speed 1,500 r.p.m. The highest head used for a reaction turbine in the United States is 900 ft. static or 849 ft. net at the Oak Grove plant of the Portland Electric Power Co. The unit runs at 514 r.p.m. and develops 35,000 hp. It was built by the Pelton Water Wheel Co.

The largest reaction turbines, so far as dimensions are concerned, are at Spier Falls in New York State. They produce 57,000 hp. at 81.8 r.p.m. under a head of 81 ft. They were built by Newport News Shipbuilding and Drydock Co. Second to them in size are the units at Conowingo on the Susquehanna River, which produce 54,000 hp. at 81.8 r.p.m. under a head of 89 ft. Part of these units were built by Allis-Chalmers Mfg. Co., and part by the I. P. Morris division of the Baldwin-Southwark Corporation. The over-all diameter of the runner is nearly 18 ft., and it weighs over 200,000 lb. A section through one of these turbines is shown in Fig. 246.

The most powerful reaction turbines are those at Boulder Dam on the Colorado River which develop 115,000 hp. per unit. The head is 525 ft. and the speed 180 r.p.m.

Propeller-type turbines are being used under heads as low as 8 and as high as 66 ft. Under the 8-ft. head there are units of 800 hp. each operating at 80 r.p.m., while under the 66-ft. head there are units of 37,500 hp. at 138.5 r.p.m. The largest size propeller turbines are at Safe Harbor on the Susquehanna River. They develop 42,000 hp. under a head of 55 ft. at 109.1 r.p.m. The outer diameter of the runner is 220 in.

The maximum efficiency attained by a Francis-type reaction turbine is 93.8 per cent, while that attained by the propeller type of turbine is 91 per cent.

CHAPTER XVII

THEORY OF THE REACTION TURBINE

191. Introductory Illustration.—The reaction turbine is so called because an important factor in its operation is the reaction of the streams of water discharged from the runner. It is well to bear in mind, however, that the total dynamic effect is due to the entire change in the momentum of the water just as in the impulse turbine.

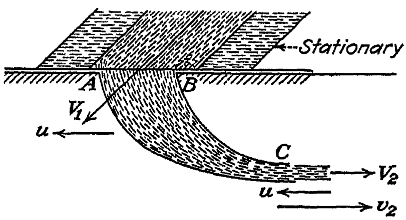


FIG. 272.

As an illustration, consider the vessel ABC of Fig. 272 into which water enters across AB with a velocity V_1 and is discharged at C with a velocity V_2. Now, the reaction of the jet alone could be determined by an application of Art. 157. But the total force is due not only to this reaction but also to the impulse of the water entering at AB. It is not feasible to separate the effect of impulse from that of reaction, neither is it necessary to do so. The horizontal component of the total dynamic force is obtained directly by

$$F = \frac{W}{g}(V_1 \cos \alpha_1 + V_2).$$

Suppose, now, that this vessel moves to the left with a uniform translation u. Assume that in some way the water is still supplied to it with a velocity V_1. This might be the case if the vessel passed under a series of stationary passages each of which in turn was permitted to discharge water into it. The value ot the absolute velocity of discharge is now

$$V_2 = v_2 - u.$$

Inserting this value above,

$$F = \frac{W}{g}(V_1 \cos \alpha_1 + v_2 - u).$$

This equation indicates that F decreases as the speed increases just as in the case of the impulse turbine. Also, the water entering the vessel at AB is under pressure and is not a free jet Therefore, V_1 must be less than $\sqrt{2gh}$. Since all the passages are completely filled with water, the equation of continuity can be applied, and it will show that V_1 and v_2 are inversely proportional to the areas of their respective streams. But the value of v_2 depends upon the losses of head, and in a real turbine these hydraulic losses vary with the speed. Since V_1 is proportional to v_2, it follows that V_1 varies with the speed of the wheel.

Thus some fundamental differences between impulse and reaction turbines are that in the former $V_1 = c_v\sqrt{2gh}$, where c_v is a velocity coefficient nearly equal to unity. This velocity, and hence the amount of water discharged by the nozzle, is entirely independent of the design of the wheel and its operation. But for the reaction turbine

$$V_1 = c\sqrt{2gh}, \tag{248}$$

where c is not a velocity coefficient but a factor whose value varies from about 0.6 to 0.8 for ordinary designs. The value of c depends upon the design of the wheel and the speed at which it is run under a given head. This means that c is also a function of ϕ, where ϕ has the meaning given by Eq. (244).

With the radial-flow type of turbine, centrifugal force also causes the value of c to vary with the speed of the wheel, the head remaining constant. The centrifugal force opposes the flow of water in the case of the inward-flow turbine so that, as the speed increases under a constant head the discharge tends to decrease as shown in Fig. 273. But there are other influences at work also, so that for some inward-flow turbines the value of q actually increases somewhat as the speed increases above zero, but after a certain speed is exceeded the rate of discharge falls off again.

192. Torque Exerted.—The preceding article merely illustrates a few fundamental points regarding the reaction turbine. Since with the real machine the radii of the water at inflow and outflow

differ materially, it is not feasible to compute the force exerted by the water, and the torque must be obtained instead. Before proceeding any further with the theory, it should be noted that, while our equations are rational, they must assume that all particles of water move in similar paths with equal velocities. Actually, average values have to be dealt with. But these average values are not known with any precision. For example, there is no assurance that the angles α_1 and β_2 are the same as the angles of the guide vanes and the runner vanes, respectively. In fact, there is some evidence to indicate that they differ by

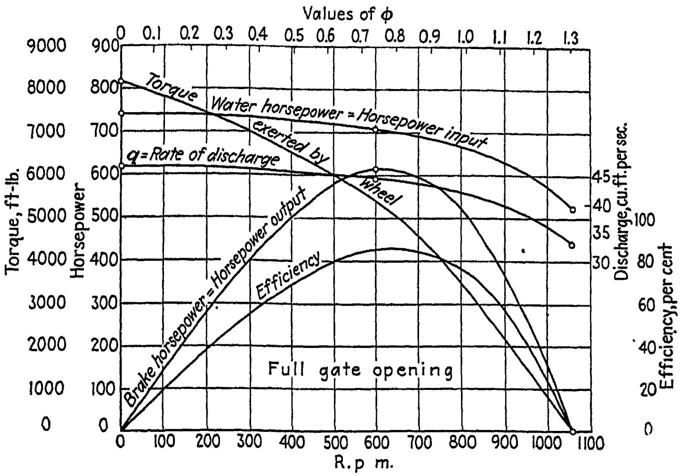


FIG. 273.—Test of 27-in. I. P. Morris turbine. Head and gate opening constant. Speed variable.[1]

as much as 5 or 10 deg. The same condition exists with regard to the areas of the streams and all other dimensions used. Thus the numerical results of such computations cannot be expected to agree precisely with actual facts.

Despite this, the theory, has its value. It serves to explain the principles of operation of such machines, to indicate the nature of their actual characteristics, and to account for numerous observed facts. In design the theory shows what proportions are desirable and what the effect of certain changes of dimensions

[1] Figures 273 and 274 are from the test of a reaction turbine in the Cornell University power plant. See DAUGHERTY, R. L., "Investigation of the Performance of a Reaction Turbine," *Trans. A. S. C. E.*, vol. 78, p. 1270, 1915.

would be. Thus with some actual test data to work from, existing designs might be altered by the theory with some degree of assurance.

In order to compute the torque exerted upon the runner by the water, the fundamental formula of Art. 164 might be taken:

$$T = \frac{W}{g}(r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2). \tag{249}$$

Just as in the case of the impulse turbine in Art. 180, the values of V_2 and α_2 are variable and unknown, and it is necessary to replace them in terms of known quantities. It is assumed that all the dimensions of the wheel and the values of V_1 and u_1 are known.

From the vector diagram it may be seen that

$$V_2 \cos \alpha_2 = u_2 + v_2 \cos \beta_2.$$

But $u_2 = (r_2/r_1)u_1$, and, since the passages are completely filled with water in the reaction turbine, the equation of continuity gives $q = A_1 V_1 = a_2 v_2$, or

$$v_2 = \left(\frac{A_1}{a_2}\right) V_1. \tag{250}$$

(Contrast this procedure with that for the impulse wheel in Art. 180 and see again Art. 169.) Then

$$V_2 \cos \alpha_2 = \frac{r_2}{r_1} u_1 + \frac{A_1}{a_2} \cos \beta_2 V_1,$$

from which the value to be used in Eq. (249) may be computed. In the actual use of Eq. (249) the value of W would have to be determined for a given value of u_1, either by experiment or by computing the rate of discharge by theory.

193. Power.—The power developed by the water is determined by multiplying T by the angular velocity. The torque actually exerted *by* the shaft and the power delivered are obtained by multiplying these values by the mechanical efficiency.

The hydraulic efficiency of the turbine is obtained by

$$e_h = \frac{T\omega}{Wh} = \frac{Wh''}{Wh} = \frac{h''}{h}.$$

It is difficult to obtain the hydraulic efficiency by test, as it is necessary to determine the bearing friction and also the disk

friction due to the drag of the runner through the water in the clearance spaces. But these losses may be allowed for, and the hydraulic efficiency then secured approximately.

Because of the necessary defects of the theory, the hydraulic efficiency may be assumed with less error than is usually involved in computing T. For turbines of rational design and running at their proper speeds, the value of the hydraulic efficiency may range from 0.80 to 0.95. The higher values are found only in large turbines and with favorable proportions. Only experience can enable one to select the proper value between these two

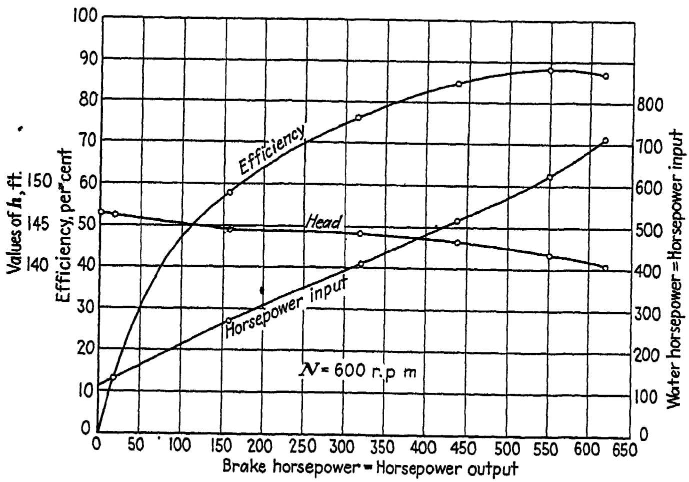


FIG. 274.—Test of 27-in. I. P. Morris turbine. Head and ϕ approximately constant. Gate opening variable.

extremes, which are not necessarily limits. For improper speeds and incorrect designs no values can be assigned.

The curves of Fig. 273 show the characteristics of a reaction turbine with a fixed gate opening and the speed variable. These are similar to those of the impulse wheel except that q is not a constant. Hence maximum power developed and the maximum efficiency do not necessarily occur at the same speed. The characteristics of the same turbine at a constant speed are shown in Fig. 274. The maximum efficiency is 88 per cent at 550 hp. under a head of 141.8 ft.

194. Speed.—Although the water flowing through the runner of a reaction turbine is entirely confined, the velocity undergoes changes similar to those in the impulse turbine, except that for a

fixed gate opening the angle α_1 is constant. Hence the values of V_2 and α_2 vary in just the same way, as is shown in Fig. 241.

The speed at which the efficiency is the highest will be somewhere in the neighborhood of the one for which the discharge loss is the least, which is when V_2 is a minimum. For this condition it will be found that approximately $\alpha_2 = 90$ deg. or that $u_2 = v_2$. Note that these two conditions are not identical, but they differ very little. It is customary to assume one or the other according to convenience.

In the case of the reaction turbine, V_1 is less than for an impulse turbine under the same head. But the water at entrance is under pressure, and, as it flows through the runner, this is converted into velocity. Hence at discharge v_2 may easily be as large as in the case of the impulse wheel. And if $u_2 = v_2$, it may be seen that u_2 will be about the same in either type. But with the inward-flow reaction turbine, u_1 is greater than u_2, and therefore the peripheral velocity of the reaction turbine is greater than that of the impulse wheel.

Not only is the peripheral speed higher for maximum efficiency, but also the runaway speed is higher. The maximum value of ϕ for the reaction turbine is about 1.30, though with some it may easily exceed this value. And for the normal speeds at which the maximum efficiency is obtained,

$$\phi_e = 0.60 \text{ to } 0.90,$$

the exact value for a given wheel depending upon its design.

The question naturally arises as to how it is possible for the peripheral velocity of the runner to be greater than the velocity of the entering water, as it often is. If the velocity of any moving object is in the same direction as that of the water, then it is obvious that none of the water can overtake it unless it is moving more slowly than the water. But if, on the other hand, its direction of motion is at right angles to that of the water, then it may be seen that water may strike it, no matter what their respective velocity may be in magnitude. Thus in Fig. 275 the amount of water that flows into the wheel may be said to be a function not merely of V_1 but rather of the normal component $V_1 \sin \alpha_1$ (which also equals $v_1 \sin \beta_1$). As long, therefore, as the angle α_1 is not 0 deg., water may flow into the wheel whatever its peripheral speed may be.

Since u_1 and V_1 are proportional to the factors ϕ and c, the former may be replaced by the latter in many cases and gain in generality and simplicity, since ϕ/c is exactly the same value as u_1/V_1. In Fig. 275 (*a*) the velocity of the water is greater than that of the wheel, but in (*c*) it is less. These three diagrams are drawn to scale for values that are typical of three different types of turbine runner.

The angle β_1 is determined by the velocity diagram. The runner vane should then be so shaped at entrance that its angle β'_1 should agree with β_1 in order that no unnecessary turbulence loss may be set up due to an abrupt change in the flow of the water. This vane angle β'_1, once being fixed by construction, is

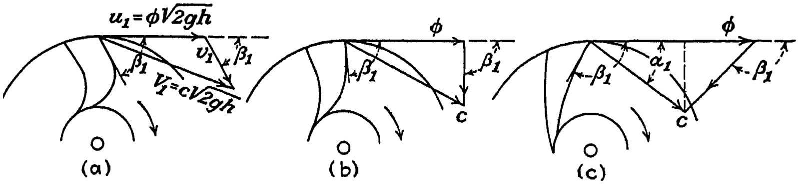


FIG. 275.

then suitable only for one ratio of ϕ to c. If the wheel runs too slowly, β'_1 will be greater than β_1. If it runs too fast, β'_1 will be less than β_1.

195. Values of c_e and ϕ_e for Maximum Efficiency.—The turbine should run normally at such a speed under any head that the maximum efficiency will be attained. This will naturally be when the total of all the losses is a minimum. While the loss of energy at discharge is not the largest source of loss, it is the one that seems to vary most with a change in speed. Hence it is customary to neglect the variation in the other losses and determine the speed at which this will be a minimum. The value of V_2 will be very close to a minimum when the angle α_2 is 90 deg. Such a discharge is called *radial* from the fact that in the original inward-flow turbine, it was directed toward the axis of rotation and hence lay along a radius. In the more modern type of mixed-flow turbine it is often called *axial,* since it is parallel to the axis. In general, it is often spoken of as *perpendicular*, since it is at right angles to the linear velocity of the runner at this same point. But the first two terms are often used regardless of the physical facts.

The speed at which $\alpha_2 = 90$ deg. is very close to that at which the maximum efficiency is actually found in the case of the Pelton wheel and the "low-speed" reaction turbine such as shown in Fig. 256 (*a*). But for the "high-speed" type, as in Fig. 256 (*b*), there is almost always a whirl to be found in the draft tube when the wheel is running at the point of maximum efficiency. This is because, for this type of turbine, other factors become of increasing importance. The mixed-flow type of runner is not amenable to any simple mathematical analysis, and it would be beyond the scope of this book to undertake to present its theory in any adequate way. Hence the following treatment is to be understood as being applicable largely to the former type of turbine runner, although it will give approximately correct results, even for the latter.

The angle that the runner vane at entrance makes with u_1 will be denoted by β'_1. As the turbine is ordinarily designed, the value of the vane angle would be such that it would agree with β_1 as determined by the vector diagram for this same speed. But at any other speed the value of β_1 would be different from β'_1; hence there would be an abrupt change in the direction of the water entering the runner, giving rise to what is known as *shock loss*.

The following expressions therefore apply *only* to the *special* case where $\alpha_2 = 90$ deg. and $\beta'_1 = \beta_1$. From the vector diagram of velocities then

$$V_1 \sin \alpha_1 = v_1 \sin \beta_1 = v_1 \sin \beta'_1.$$
$$V_1 \cos \alpha_1 = u_1 + v_1 \cos \beta_1 = u_1 + v_1 \cos \beta'_1.$$

Eliminating v_1 between these two equations,

$$u_1 = \frac{\sin (\beta'_1 - \alpha_1)}{\sin \beta'_1} V_1 \tag{251}$$

as the relation between u_1 and V_1 when there is no loss at entrance to the runner.

The power delivered by the water to the runner may be expressed as

$$T\omega = Wh'' = e_h Wh,$$

where T has the value given by Eq. (249). Thus

$$Wh'' = \frac{W}{g}(r_1 V_1 \cos \alpha_1 - r_2 V_2 \cos \alpha_2)\frac{u}{r}.$$

If the discharge is radial, $\alpha_2 = 90$ deg., and hence $V_2 \cos \alpha_2 = 0$. Therefore,

$$h'' = e_h h = \frac{u_1 V_1 \cos \alpha_1}{g}. \tag{252}$$

Solving Eqs. (251) and (252) simultaneously, we have

$$V_1 = \sqrt{\frac{e_h 2gh}{2} \frac{\sin \beta'_1}{\sin (\beta'_1 - \alpha_1) \cos \alpha_1}}.$$

$$u_1 = \sqrt{\frac{e_h 2gh}{2} \frac{\sin (\beta'_1 - \alpha_1)}{\sin \beta'_1 \cos \alpha_1}}.$$

From this it follows that

$$c_e = \sqrt{\frac{e_h \sin \beta'_1}{2 \sin (\beta'_1 - \alpha_1) \cos \alpha_1}}. \tag{253}$$

$$\phi_e = \sqrt{\frac{e_h \sin (\beta'_1 - \alpha_1)}{2 \sin \beta'_1 \cos \alpha_1}}. \tag{254}$$

It must be borne in mind that Eqs. (253) and (254) can be applied only for the special case stated. For any other speed a different procedure would be necessary, but it will not be given here.[1]

The speed desired is the one for which the gross efficiency is a maximum, and this may not be quite the same as the one for which the hydraulic efficiency is the highest. Hence the true value of ϕ_e may differ slightly from the value given by Eq. (254).

These equations appear to be independent of conditions at outflow from the runner. But it must be noted that they are to be used only upon the assumption that the dimensions used at exit will be such as to make $\alpha_2 = 90$ deg., when c_e and ϕ_e have the values given.

An interesting result may be obtained by multiplying Eqs. (253) and (254). This gives

$$c_e \phi_e = \frac{e_h}{2 \cos \alpha_1}. \tag{255}$$

This would indicate that, all other things being equal, the higher the value of ϕ_e the smaller that of c_e. With the impulse turbine, to which these equations apply also, ϕ_e is small, but $c = c_v$ and

[1] A general relation between c and ϕ for all conditions will be found in the author's "Hydraulic Turbines," McGraw-Hill Book Company, Inc., 1920.

is near unity. With the reaction turbine, ϕ_e is larger than for the impulse wheel, but c_e is smaller.

196. Theory of the Draft Tube.—If the draft tube is properly designed, its area next to the runner should be such that the velocity in it is equal to V_2, the absolute velocity of discharge; otherwise there will be an abrupt change of velocity, involving losses. For Fig. 276 may be written

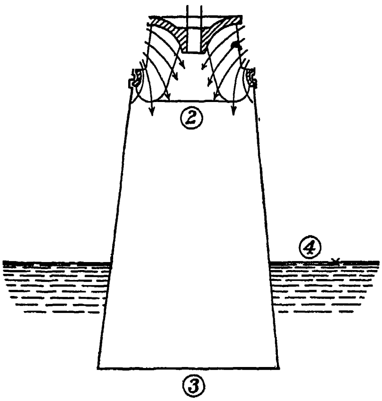


FIG. 276.

$$H_2 = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g},$$

and

$$H_4 = 0.$$

The losses between points (2) and (4) are made up of the friction losses within the tube, h_f, and the discharge loss at (3). Applying the general equation between points (2) and (4), we have

$$\frac{p_2}{w} = -z_2 - \frac{V_2^2}{2g} + h_f + \frac{V_3^2}{2g}. \tag{256}$$

The larger the diameter of the tube at its mouth the less will be the value of V_3, and hence the less the pressure at the point of discharge from the runner. But if too great a rate of diffusion is provided for in the tube, the flow in it will be unstable, and the friction loss h_f will be increased, as shown in Art. 113. The pressure at the top of the draft tube should not be made less than about 5 ft. absolute, and the value of z_2 determined accordingly. A high-speed turbine runner with a large value of V_2 cannot be set so far above the water level as a low-speed turbine with a smaller value of V_2.

In the ideal case, the energy or head saved by a diverging draft tube would be

$$\frac{V_2^2}{2g} - \frac{V_3^2}{2g} = \left[1 - \left(\frac{A_2}{A_3}\right)^2\right]\frac{V_2^2}{2g},$$

where V_2 is the axial component of velocity. Owing to friction losses within the tube, the actual saving is less. If the type

of draft tube shown in Fig. 265 is used, the whirl component can be utilized also, as explained in Art. 57.

197. Head on Reaction Turbine.—For a reaction turbine the draft tube is an integral part of the machine; hence (Fig. 277) the head under which it operates is given by

$$H_A - h_f = H_C - H_F = \frac{p_C}{w} + z_C + \frac{V_C^2}{2g} \tag{257}$$

This is the value of h upon which computations are based, and it is the one to be used in determining the efficiency of the turbine.

However, though the turbine maker usually constructs or designs the draft tube also, he is often limited by the conditions of the setting and may not be able to use the proper proportions.

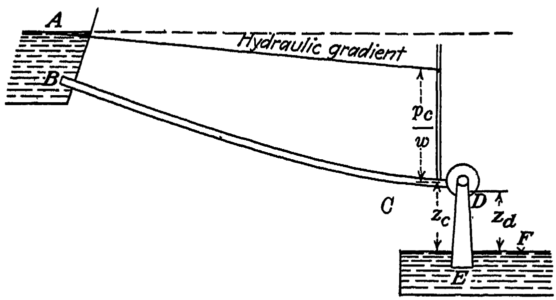


FIG. 277.

In order to eliminate this defect in the setting, over which he has no control, the velocity head at E is sometimes deducted from the value given by Eq. (257). If it were feasible to eliminate the friction in the draft tube as well, then there would remain the efficiency of the runner alone, which is independent of the draft tube. But what is usually desired is the efficiency of the entire unit from the intake of the casing to the tailrace.

198. PROBLEMS

349. A certain reaction turbine was found by actual test to have a hydraulic efficiency of 0.83 when $\phi = 0.670$ and $c = 0.655$. The angles were $\alpha_1 = 13$ deg. and $\beta'_1 = 115$ deg. Compute the values of ϕ_e and c_e and compare with the actual values. (The slight discrepancy between the two is largely due to the fact that shockless entrance and radial discharge were not obtained at exactly the same speed.) *Ans.* $\phi_e = 0.678$, $c_e = 0.628$.

350. In the test of the Cornell University turbine the pressure was read by a mercury manometer attached near the intake flange where the diameter was 30 in. At full load when the discharge was 44.5 cu. ft. per sec., the

manometer read 9.541 ft. of mercury, the top of the shorter mercury column being 0.500 ft. above the intake. If the elevation of the intake above the water level in the tailrace is 9.230 ft., find the head on the turbine.

Ans. 140.5 ft.

351. In the turbine of Prob. 350 the diameter of the draft tube at the upper end is 24.5 in. and at the bottom 42 in. Find the gain in head due to its use when the discharge is 44.5 cu. ft. per sec. *Ans.* 2.54 ft.

352. The top of the draft tube in Prob. 351 is 10.0 ft. above the level of the water in the tailrace. Neglecting the friction in it but considering the discharge loss at the bottom, find the pressure at its top. What would it be if the diameter were uniform? *Ans.* -12.54 ft., -10 ft.

353. A reaction turbine by test was found to discharge 31.8 cu. ft. of water per sec. when running at 600 r.p.m. under a head of 143.1 ft. If $A_1 =$ 0.535 sq. ft. and $D = 27$ in., find values of c and ϕ.

Ans. $c = 0.619$, $\phi = 0.737$.

354. Prove that for a reaction turbine the torque exerted by the water is given by

$$T = \left(\frac{W}{g}\right)r_1\left[\left(\cos\alpha_1 - \left(\frac{r_2}{r_1}\right)\left(\frac{A_1}{a_2}\right)\cos\beta_2\right)V_1 - \left(\frac{r_2}{r_1}\right)^2 u_1\right].$$

355. For a reaction turbine the dimensions are $\alpha_1 = 35$ deg., $\beta'_1 = 136$ deg., $e_h = 0.845$. Compute the values of ϕ_e and c_e.

Ans. $\phi_e = 0.85$, $c_e = 0.60$.

356. A reaction turbine under a head of 300 ft. has the following dimensions; $\beta_2 = 162$ deg., $A_1 = 1.074$ sq. ft., $a_2 = 1.552$ sq. ft., $r_1 = 2.0$ ft., $r_2 = 1.6$ ft., and for maximum efficiency $\phi_e = 0.625$ and $c = 0.759$. What is the power lost at discharge from runner? (Note that for the conditions given $\alpha_2 = 90$ deg.) *Ans.* 101.5 hp.

357. For a reaction turbine with $\alpha_1 = 18$ deg., $\beta_2 = 165$ deg., $r_1 = 2.0$ ft., $r_2 = 2.5$ ft., $A_1 = 6.80$ sq. ft., $a_2 = 7.13$ sq. ft., $k = 0.2$, $h = 70$ ft., $q = 356$ sec. ft., $N = 184$ r.p.m., find the head utilized and the hydraulic efficiency. *Ans.* 85 per cent.

358. Find the torque and the power developed in runner in the foregoing.

Ans. 2,410 hp.

359. What should be the vane angle at entrance to the runner in Prob. 356 and the area a_1? What is the difference in pressure at entrance to, and outflow from, the runner?

CHAPTER XVIII

TURBINE LAWS AND FACTORS

199. Operation under Different Heads.—In the entire discussion in the two preceding chapters it has been assumed that the head remained constant though the other quantities might vary. But a turbine may be installed in a plant where the head changes from time to time, and also a given design of turbine might be used in different plants under a wide range of heads. Thus it is desired to investigate this phase.

Let us recall the expression $u_1 = \phi\sqrt{2gh}$. Suppose, now, that a turbine is compelled to run at a constant speed while the head varies. It is clear that ϕ also varies, then, just as it would in the preceding case. But it would be possible under some circumstances to change the speed as well in such a way as to keep ϕ a constant. Hence it is necessary to consider two distinct cases when the head changes; one is where ϕ is constant, and the other is where ϕ also changes.

If ϕ remains constant, the wheel speed must vary as $\sqrt{h}$. But a definite value of ϕ is accompanied by a definite value of c. Hence the rate of discharge must also vary as $\sqrt{h}$, since $V_1 = c\sqrt{2gh}$. Now, the power of the water is proportional to the product of q and h. Since q varies as $\sqrt{h}$, it follows that the power varies as $h^{3/2}$. And in similar fashion it may be shown that the torque varies as h.

The hydraulic efficiency is a function of c, ϕ, and the turbine dimensions. As long as ϕ remains constant, the hydraulic efficiency remains the same regardless of the head. This must be true because the hydraulic losses may all be shown to vary as $h^{3/2}$, just as the power input. But the friction of the bearings and the windage or the disk friction of the runner do not vary in the same way. It is not possible to formulate an exact law for this, but they may be said to vary between N and N^2. Since N varies as $\sqrt{h}$, they must vary between $h^{1/2}$ and h. Hence the

mechanical losses become a smaller percentage of the total as the head increases.[1] But, except for very low heads, the difference in the efficiency is usually a matter of not more than 2 or 3 per cent at most (see Fig. 278).

Now, if the speed remains constant while the head changes, or if it does not vary as $\sqrt{h}$, the value of ϕ will change. Referring to Fig. 273, it may be seen that this means a change of c also. Hence the efficiency will change. Thus none of the simple proportions that have just been stated will be true in such a

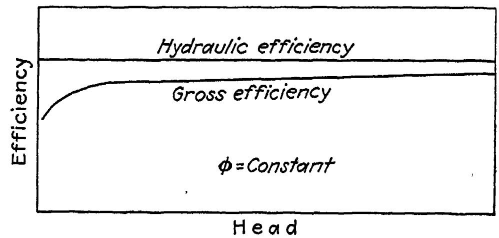


FIG. 278.—Effect of head upon efficiency of a given turbine.

case. It is impossible to calculate the new results unless curves, such as those of Fig. 273, are available, or unless some complex equations are used which will give values of all these quantities for any value of ϕ.

200. Different Sizes of Runner.—If a series of runners are all built of the same design with the same angles and proportions so that one is simply an enlargement or reduction of another, they should all have the same values of ϕ_e and c_e. Since their peripheral speeds would all be the same under a given head, it follows that their rotative speeds would be inversely proportional to their diameters. And the area A_1 would be proportional to D^2. Hence their capacity and power would vary as D^2. Thus if the performance of one runner is known, that of the rest of the series may be predicted with some assurance, due allowance being made for slight increases in efficiency as the size increases.

[1] An impulse wheel should be set with sufficient space on either side of the buckets at discharge so that the water rebounding from the walls will not strike them. The velocity with which the water rebounds is proportional to V_2 and hence to $\sqrt{h}$. Therefore, if this space is not ample for all values of h, a point may be reached where this action would decrease the efficiency.

Since

$$u_1 = \phi\sqrt{2gh},$$

and also

$$u_1 = \frac{2\pi\, r_1 N}{60} = \frac{\pi DN}{12 \times 60},$$

then

$$\phi = \frac{\pi DN}{12 \times 60 \times \sqrt{2g} \times \sqrt{h}}.$$

From this

$$\phi = \frac{DN}{1{,}840\sqrt{h}},$$

or

$$N = \frac{1{,}840\phi\sqrt{h}}{D}, \tag{258}$$

where ϕ may have any value. But if ϕ_e is used, the result is N_e, the speed at which the maximum efficiency is attained. For the two types of turbines in common use:

Impulse wheel $\phi_e = 0.43$ to 0.48.
Reaction turbine $\phi_e = 0.60$ to 0.90,

according to design. And as to capacity,

$$q = K_1 D^2 \sqrt{h} \tag{259}$$

where K_1 has the following range of values:

Impulse wheel $K_1 = 0.0002$ to 0.0005.
Reaction turbine $K_1 = 0.0014$ to 0.0360.

It must be understood that these constants are based upon values corresponding to ϕ_e and that the speed of the wheel must be such that ϕ_e is obtained if they are to apply.

Making a suitable allowance for the efficiency, the power delivered by the turbines can be determined when the discharge is known.

It may be seen that the peripheral speed of the reaction turbine is higher than that of an impulse wheel and that it may be varied through a wider range by changes in the design. Also, the values of K_1 show that for a given diameter a reaction turbine can discharge more water, and hence develop more power, than an impulse turbine. That means that if they are to deliver the same

power, the diameter of a reaction turbine will be much less than that of a corresponding impulse wheel. Thus for a given head and power the rotative speed of the reaction turbine will be much higher than that of the impulse wheel, owing both to its higher peripheral speed and to its much smaller size.

201. Specific Speed.—A useful factor in turbine work is one that will now be derived. It involves the head, speed, power, and efficiency.

Since power is proportional to D^2 and to $h^{3/2}$, b.hp. $= K_2D^2h^{3/2}$. This may be rewritten as

$$\sqrt{K_2}D = \frac{\sqrt{\text{b.hp.}}}{h^{3/4}}.$$

Inserting the value of D as given by Eq. (258), then

$$\sqrt{K_2}\frac{1{,}840\phi_e\sqrt{h}}{N_e} = \frac{\sqrt{\text{b.hp.}}}{h^{3/4}}.$$

Rearranging this and letting $\sqrt{K_2}\ 1{,}840\phi_e = N_s$, then

$$N_s = \frac{N_e\sqrt{\text{b.hp.}}}{h^{5/4}*}. \qquad (260)$$

While any value of N might be used, the expression has but little meaning unless a particular value is employed. That is generally understood to be N_e, the value of N at which the maximum efficiency under a given head is attained. As to the value of brake horsepower, it should logically be the one for which the maximum efficiency is obtained under the given head. But in some cases the value of the maximum power at this same speed is used. The physical significance of this factor is that if a turbine of a given design is enlarged or reduced in size so that it will develop 1 hp. under 1 ft. head, then N_s is the value of its r.p.m. under those circumstances.

The quantity N_s is generally known as *specific speed.* Other names applied to it are unit speed, type characteristic, and characteristic speed. Its value indicates the class to which a turbine belongs. Thus it has been seen that for a given head and power the impulse wheel runs at a relatively low r.p.m. Therefore it would have a low value of N_s.

* $h^{5/4} = h \times h^{1/4} = h\sqrt{\sqrt{h}}.$

For an impulse wheel under a given head at a given speed the power would increase with the size of the nozzle used. Thus there need not be any lower limit to the value of N_s, but the upper limit would be the one for which the maximum-size jet that could be employed was obtained. It is found that the efficiency is not appreciably reduced until after a value is passed of about 4.5 for N_s and that after a value of 6 the jet is so large for the size of the wheel that the efficiency drops off materially. But any value above 4.5 involves some sacrifice of efficiency.

For the reaction turbine there are limits in both directions, as indicated below, though these may be extended in future designs. The values of the specific speed are

Impulse wheel $N_s = 0$ to 4.5 (6 max.).
Reaction turbine $N_s = 10$ to 100.
Propeller type $N_s = 80$ to 200.

For a given turbine the value of N_s is naturally a constant, but it is also practically constant for a whole series of runners of the same design regardless of size. The larger the diameter of a runner the greater its power, but the less the value of N for a given head. Hence the product remains constant.

Values of N_s given for the impulse wheel are for a single jet upon a single wheel. When two or more jets are used the power is naturally increased without changing the speed. This enables values between 6 and 10 to be obtained, if necessary. For values above 200 the conditions are impossible with present practice. Either the power or the speed of the unit must be decreased.

The specific speed factor shows that the impulse wheel is a low-speed, low-capacity turbine and the reaction turbine is a high-speed, high-capacity turbine. The use of these words is relative rather than absolute. Thus the turbine in Fig. 252 runs at only 55.6 r.p.m., but its specific speed is 82.3, thus indicating that it is a high-speed wheel. For the speed is high as compared with that of other turbines of the same power under that head. For instance, the speed of an impulse wheel for similar conditions would be only 4 r.p.m.

202. Uses of Specific Speed.—The values of N_s, as of all other factors in this chapter, are supposed to be obtained from test data, not computed by theory. They serve to classify a turbine and indicate to what type it belongs. They are useful in selecting units for a prospective plant. For such a case the head is known,

but the size and speed of the units are not. If it is desired to use wheels of a certain type, that fixes the value of N_s between narrow limits, and it is easy to compute the combinations of speed and power that can be produced. Or, if the speed and power are fixed, at once the type of turbine required may be found.

203. Factors Affecting Efficiency.—The efficiency of the impulse wheel is practically independent of the size of the wheel. The author makes this statement after testing sizes from 12 to 84 in. in diameter and comparing all the other test data that are accessible. It would seem reasonable that this should be so, for there is no loss in connection with the impulse wheel that would not vary in proportion to the power of the wheel. Aside

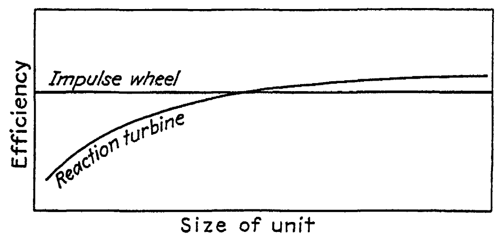


FIG. 279.—Effect of size of turbine upon its efficiency.

from questions of design and workmanship, the efficiency would appear to be a function of the specific speed. Too low a value of the specific speed would mean a large diameter of wheel for a given power output with a consequently large friction and windage loss. Too high a specific speed would mean that the jet was too large for the wheel and buckets, with a consequent lowering of the hydraulic efficiency. The most favorable value of N_s is about 4.0, and the best efficiency that has been obtained is about 88 per cent, but values around 82 per cent are more common.

With the reaction turbine the efficiency is a function of its size. This is partly due to the fact that the hydraulic efficiency increases with the size but more to the fact that the volumetric efficiency increases. With a reaction turbine there is always a certain amount of leakage between the guides and the runner, so that a portion of the water escapes through the clearance spaces and does not pass through the wheel. The area of these clearance rings would naturally be less in proportion to the area of the wheel passages as the size of the wheel increases. Hence a much larger percentage of the water is made to deliver its power

to the runner. Such a condition does not exist with the impulse wheel. This leads to comparative values for the two types, as shown in Fig. 279.

Another distinction between the two types of turbines is that the reaction turbine suffers certain hydraulic losses on part gate that are lacking in the other. Hence, although in some cases the maximum efficiency of a reaction turbine is greater than that of the impulse wheel, the efficiency on a light load might not be so good.

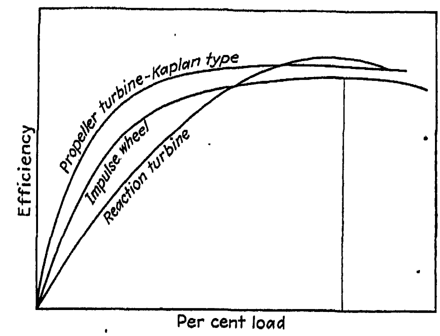


FIG. 280.

The propeller turbine with fixed blades is even lower in efficiency on part load than the reaction turbine, but the Kaplan propeller turbine with adjustable blades will give an efficiency curve that is superior on part load to that of any other type of turbine. This is because the blade angles are adjusted to avoid the shock loss encountered in the reaction turbine whenever either the speed or the load departs from the design values. The comparison of the three is shown in Fig. 280.

Like that of the impulse turbine, the efficiency of the reaction turbine also depends upon the specific speed, being less at either extreme. The best efficiencies are obtained with values of N_s ranging from 30 to 60. The efficiency of a turbine of good design and workmanship depends upon size, specific speed, and other factors to such an extent that definite values cannot be given, but for fair-size units it should range from 80 to 90 per cent and occasionally more. For small wheels, especially with unfavorable specific speeds, a value of from 60 to 80 per cent is all that should be expected.

204. PROBLEMS

360. The turbine, whose performance is shown in Fig. 274, developed its maximum efficiency of 88.0 per cent when delivering 550 hp. at 600 r.p.m. under a head of 141.8 ft. The water consumed was 38.8 cu. ft. per sec. What would be its proper speed under a head of 283.6 ft.? What would then be the rate of discharge and the horsepower?

Ans. 848 r.p.m., 54.8 cu. ft. per sec., 1,557 hp.

361. In Fig. 273 the turbine delivered 617 hp. when running at 600 r.p.m. under a head of 140.5 ft., the rate of discharge being 44.5 cu. ft. per sec., and the efficiency 87.0 per cent. If the speed is maintained at 600 r.p.m. when the head is 70.2 ft., find values of discharge, power delivered, and efficiency. (NOTE.—This can be determined only by making use of the curves for this particular turbine. The procedure would be to find the value of ϕ for the new conditions and then take values of q, hp., and e from the curves. These quantities would then have to be reduced to the proper values for the new head.) *Ans.* 29 cu. ft. per sec., 159 hp., 0.68.

362. It is desired to develop 6,000 hp. at 514 r.p.m. under a head of 625 ft. Will an impulse or a reaction turbine be required? (This can be determined by computing the specific speed.) If only 900 hp. is to be developed for the conditions given, what type of turbine will be required? It is desired to use a type of turbine whose specific speed is 30 to deliver 100 hp. under a head of 100 ft. What will be the proper r.p.m. for the unit?

Ans. $N_s = 12.75$ r.p.m., $N_s = 4.94$ r.p.m., $N_e = 948$ r.p.m.

363. (*a*) An impulse wheel is to be used for 625 hp. under a head of 144 ft. What will be the maximum rotative speed at which it can run without material sacrifice of efficiency? What will be the approximate diameter of the wheel? (*b*) What would be the minimum speed for a reaction turbine for the conditions? If $\phi = 0.60$, what would be the diameter of the runner? (*c*) What would be the maximum speed for a reaction turbine? Assuming $\phi = 0.85$, what would be the diameter of the runner? *Ans.* (*a*) 90 r.p.m., 118 in. (*b*) 199 r.p.m., 66.4 in. (*c*) 1,990 r.p.m., 9.43 in.

364. The runner in the Cornell University turbine is 27 in. in diameter. The wheel develops 550 hp. when running at 600 r.p.m. under a head of 141.8 ft. What would be the speed and power of a 54-in. runner of the same type under the same head? Would the specific speed of these two be the same? *Ans.* $N = 300$ r.p.m., 2,200 hp.

365. It is desired to use a reaction turbine with a specific speed of 82.3 in a plant where 12,000 hp. is to be developed under a head of 50 ft. What would be the r.p.m. if this power were to be developed in one, two, four, or six units? *Ans.* 100, 141, 200, 245 r.p.m.

366. It is desired to develop 20,000 hp. at 360 r.p.m. under a head of 480 ft. (*a*) Will an impulse wheel or a low-, medium-, or high-speed reaction turbine be required? (*b*) What will be the approximate diameter of this wheel? (*c*) If this same wheel is used under a head of 120 ft., what would be its r.p.m. and power? *Ans.* (*b*) 78 in. approximately.

367. A Pelton wheel is to be used under a head of 1,200 ft. where 60 cu. ft. of water per sec. is available. What should be the approximate diameter and r.p.m.? *Ans.* 76 in.

CHAPTER XIX

WATER POWER PLANTS

205. Elements of a Water Power Plant.—A complete water power development may comprise a great deal of construction and equipment aside from the powerhouse and contents—so

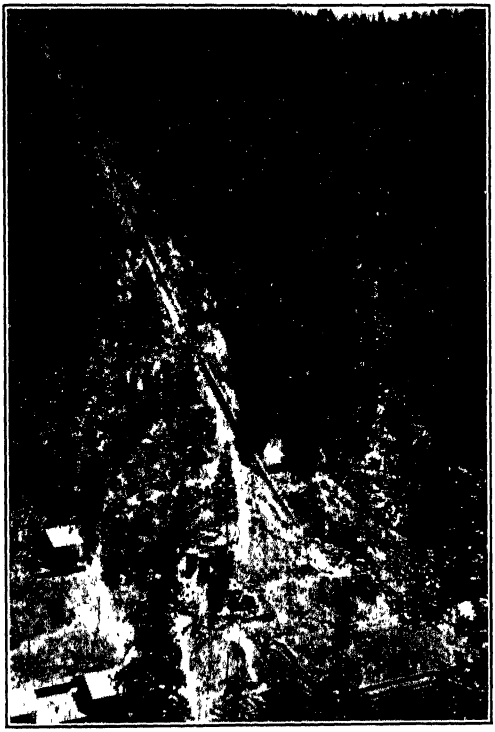


FIG. 281.—Penstock leading to Drum powerhouse of Pacific Gas & Elec. Co. under 1,375 ft. head.

much so that the cost of the latter is often a small proportion of the total investment. For a complete plant some or all of the following details may be required according to the physical situation.

A dam of some sort is usually essential. It may be nothing more than a wing wall extending a short way into the river to divert a small portion of the flow, or it may extend clear across the stream. In the latter case the water level will be raised above its former height, and also a certain amount of water will be stored up by it. If the contour of the land permits, a dam may create an artificial lake or storage reservoir. In some cases the power plant draws water directly from this body, and in others it would be used merely as a "feeder."

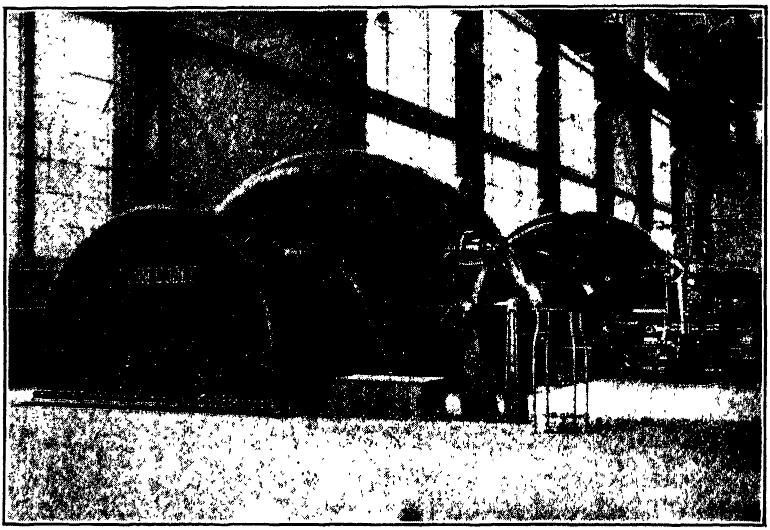


FIG. 282.—Pelton impulse wheels in Drum powerhouse of Pacific Gas & Elec. Co. h = 1,375′ static or 1,300′ under normal load; N = 360; hp. = 8,500 per wheel.

The water is conducted to the powerhouse through canals, flumes, pressure tunnels, or pipe lines, as the case may be. It is not uncommon for it to be carried from 5 to 10 miles or more in order to permit the utilization of a higher fall than could be obtained near the intake. It is desirable that the water be kept at as high an elevation as possible during the first portion of its course, as this permits the use of open channels or low-pressure pipes, which is cheaper than if the water had to be carried under high pressure all the way. This portion of the conduit is often called the *flow line* from the fact that its main function is to deliver water and not to transmit pressure.

At the end of such a flow line the water will be abruptly dropped down the hillside, as shown in Fig. 281. This portion of the pipe line is the penstock.

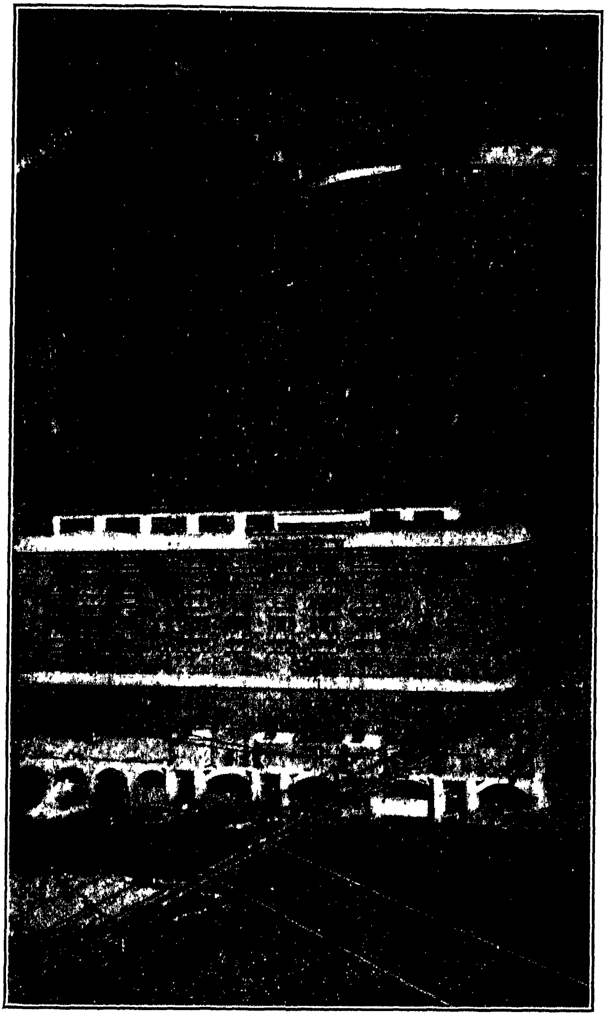


Fig. 283.—Las Plumas plant at Big Bend on the Feather River in California, containing six reaction turbines of 18,000 hp. each under a head of 465 ft. (*From a photograph by F. H. Fowler.*)

Where the distance from the intake to the power plant is a number of miles, it is desirable that there be some break in the continuity of flow, on account of speed regulation. If conditions

permit, a forebay may be constructed at the head of the penstock. The forebay is a reservoir of limited capacity whose function is to equalize the flow. Into it the water may be delivered at a uniform rate, while from it the water may be drawn by the penstock at varying rates according to the demands upon the tur-

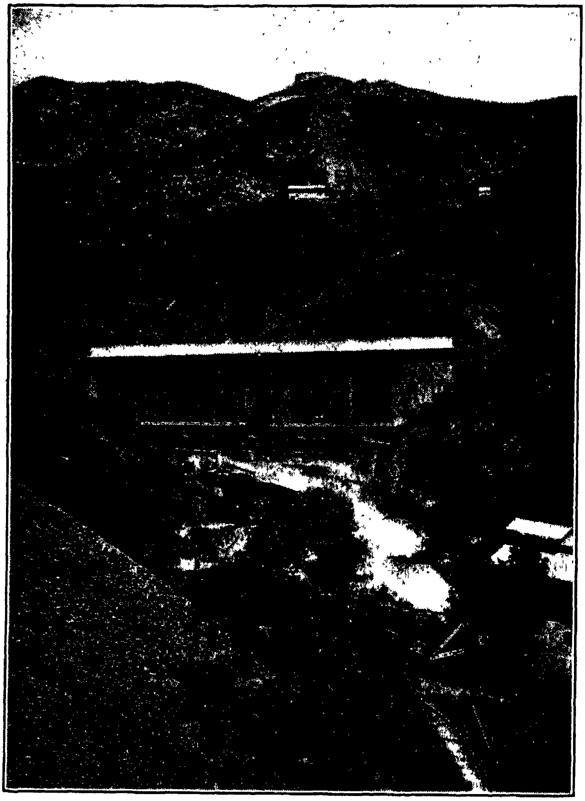


FIG. 284.—San Francisquito Power Plant No. 1 on the Los Angeles Aqueduct. Static head from maximum water level in surge chamber on crest of hill to the nozzles is 941 ft.

bines. Thus the fluctuations in the flow of water through the turbines need not extend back all the way to the source.

Where a forebay is impossible or not really necessary, it is desirable to provide surge chambers or other means of relieving the abnormal conditions attendant upon changes of flow. In the upper left-hand corner of Fig. 283 are seen a small surge chamber and an overflow. The five penstocks receive water from a

pressure tunnel 3 miles in length. In case of a sudden decrease in discharge through the turbines, the excess water could surge up the large pipe line running up the hillside; and if the surge were great enough some water would overflow, thus preventing any excessive increase in pressure.

In Fig. 284 is shown a power plant with a large surge chamber at the end of a pressure tunnel which is 7.76 miles in length. It is 100 ft. in diameter at the top, and the maximum water level is 150 ft. above the pressure tunnel. Only 35 ft. projects above

FIG. 285.—Big Creek development of Southern California Edison Company. The fall from the lake to the first powerhouse is 2,100 ft., and from that to the second powerhouse is 1,900 ft. (*Courtesy of Stone and Webster.*)

the ground. This is also provided with a spillway so that it may overflow if the surge is violent.

The water from a power plant may be discharged directly into a natural stream, or it may be necessary to construct an artificial channel for a tailrace, as in Fig. 287. In other cases, as with some of the plants at Niagara Falls, the tailrace may be a long tunnel.

206. High-head Plants.—It is impossible to establish any definite number of feet that is required to differentiate a high-head from a medium- or low-head plant. A high-head is one of several hundred feet or more, while a low-head plant would doubtless be under 50 ft. But one type shades very gradually into the other.

FIG. 286.—Power Plant No. 1 at Big Creek near Fresno, Calif. Static head = 2,100 ft. (*Courtesy of Stone and Webster.*)

In Fig. 285 is shown a high-head development, where a fall of 4,000 ft. is divided between two powerhouses in series. In this one view may be seen a complete plant with many of the features that have been described, except that a forebay is not required. The mountain ranges, which rise to a height of 11,000 ft., provide a watershed, the runoff from which is gathered by a lake about 4 miles long and with an elevation of 6,000 ft. The lake is created by the erection of three dams which can be seen placed in gaps in the hills. From the lake the water flows down the penstocks to the first plant. The discharge from this, supplemented by water from a little stream, then flows through a tunnel for a way until

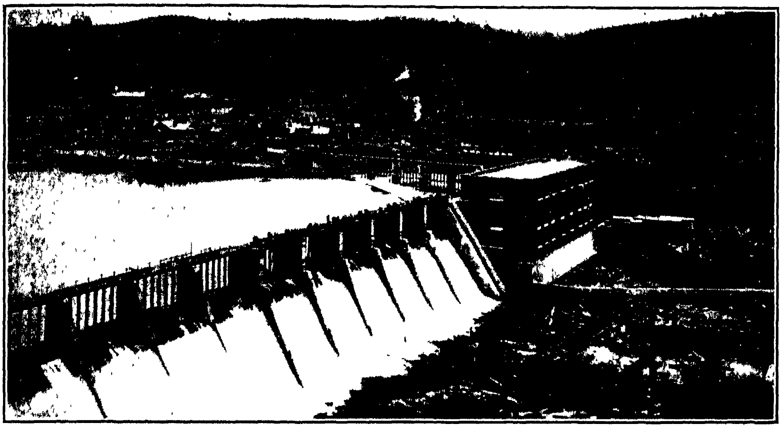


FIG. 287.—Appalachian Power Company development No. 2. Head = 49 ft., four turbines of 6,000 hp. each at 116 r.p.m. (*Courtesy of I. P. Morris Co.*)

it takes another drop to the second powerhouse which can be seen in the lower part of the picture, a little to the left of the center. In Fig. 286 is a closer view of the first plant. At the upper right-hand corner of Fig. 286 may be seen two standpipes, the water level in which will be nearly as high as that in the lake, so that the entire 2,100-ft. drop is shown here.

A high-head plant requires but little water for a given amount of power, and it is usually so situated that a storage reservoir is to be had. Consequently, it may be able to run for a long time merely on the water that is conserved by the creation of such reservoirs. It is always necessary to have a penstock and many of the other details that have been enumerated.

207. Low-head Plants.—A typical low-head plant is shown in Fig. 287. The head, under which the turbines operate, has been

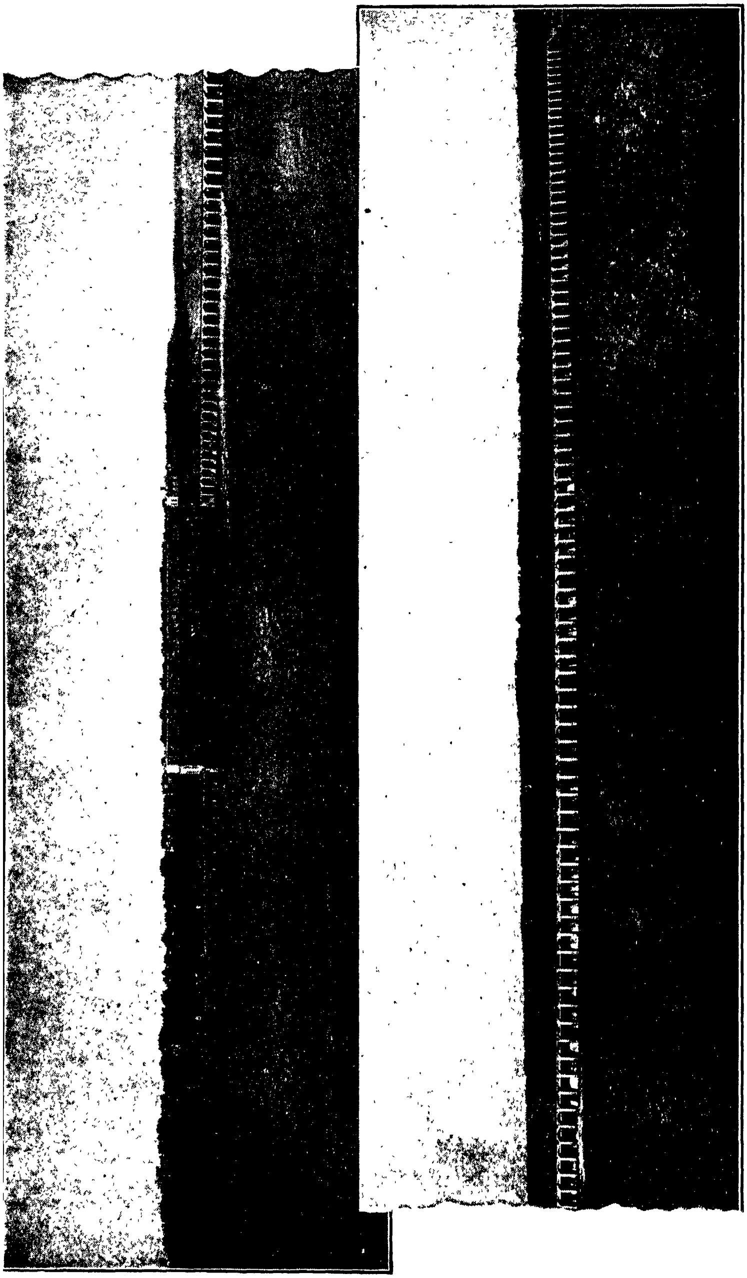


FIG. 288.—The Mississippi River Power Company at Keokuk, Ia. Head = 32 ft., capacity is 15 units of 10,000 hp. each at 57.7 r.p.m. Maximum capacity of plant is 200,000 hp. (*Courtesy of Mississippi River Power Co.*)

practically created by the erection of a dam. There are no pipe lines, and the body of water produced by the dam now becomes the forebay. The turbines in such a plant may have any one of the three types of settings shown in Figs. 266, 267, and 268.

It may be seen that fluctuations in the flow of the river, with consequent changes in water level, cause variations in the head under which the turbines operate. This is something that scarcely exists in a high-head plant. Also, low heads are usually found in fairly flat countries, where the nature of the topography renders it impractical to store up large quantities of water, and, furthermore, under a low head a large amount of water is required to develop a given power. This makes it impossible to run very long on storage, and hence the plant is dependent upon a regular stream flow.

The differences between the high- and low-head plants are such as to require turbines of different characteristics in order to meet the conditions most satisfactorily.[1]

Another typical low-head plant is shown in Fig. 288. The length of the dam across the river is nearly a mile. While a low-head plant is often free from many of the items that are required in a high-head development, it must be remembered that it must be built to handle large volumes of water and that much massive construction is required.

208. PROBLEMS

368. A Pelton wheel is installed under a head of 500 ft. The water is delivered through 1,200 ft. of 18-in. riveted-steel pipe; the nozzle gives a 4-in. jet and has a velocity coefficient of 0.97; the water leaving the power plant flows over a suppressed rectangular weir with a very deep channel of approach. (*a*) What is the rate of discharge? (*b*) What is the pressure in the pipe near the nozzle? (*c*) What is the horsepower at this point? (*d*) What is the horsepower of the jet? (*e*) What is the height of water flowing over the weir whose width is 3 ft.? *Ans.* (*d*) 758 hp.

369. In the preceding problem what should be the peripheral speed of the wheel? (*b*) What is the value of the force exerted on the wheel if α_2 is assumed to equal 90 deg.? (*c*) What is the power output if the mechanical efficiency equals 0.98? (*d*) What is the probable r.p.m.?

Ans. 78.8 ft. per sec., 4,900 lb., 688 b.hp., 383 r.p.m.

370. There is an available water supply of 300 cu. ft. per sec. at an elevation of 1,200 ft. above a powerhouse. This water is carried by a canal of trapezoidal cross section excavated in earth and with sides that slope so that the horizontal distance is twice the vertical. The depth of water in the canal

[1] DAUGHERTY, R. L., "Hydraulic Turbines," Chap. XIII.

is to be 5 ft., and the velocity 2 ft. per sec. At the end of 4 miles the water enters a rectangular flume of smooth timber with a slope of 0.4 ft. per 1,000. This flume conducts the water 2 miles to a riveted-steel penstock 7,000 ft. long down which it flows to the powerhouse.

Actually, the penstock will decrease in diameter, and its thickness will increase as the head increases, but for the present it will be assumed to be of a uniform diameter, and its cost may be expressed as Kd^2. In this particular case a 62-in. penstock would weigh 3,000,000 lb. and cost \$150,000, which fixes the value of K which may be assumed to be constant for other diameters not too widely different. Fixed charges will be taken as 10 per cent.

Find the width of the earth canal at the bottom. Considering it to be classed as a large earth canal in good condition, find the elevation of the water surface at the junction with the flume.

Find the dimensions of the flume, assuming Kutters $n = 0.0118$, solving by trial to the nearest tenth of a foot.

Find the elevation of the water surface at the intake to the penstock. Then, assuming the value of a horsepower to be \$20 per year, determine the most economic size of penstock, trying sizes of 70, 80, 90, 100 in. Compute efficiency of resulting penstock and power delivered to the powerhouse.

Ans. Width, 20 ft., 1,196.58 ft.; flume, 5.5 by 11 ft., 1,192.35 ft.

CHAPTER XX

THE CENTRIFUGAL PUMP

209. Definition.—Centrifugal pumps are so called because of the fact that centrifugal force or the variation of pressure due to rotation is an important factor in their operation.

In brief, the centrifugal pump consists of an impeller rotating within a case, as shown in Fig. 289. Water enters the impeller at the center, flows radially outward, and is discharged around the circumference into the case. During flow through the impeller the water has received energy from the vanes resulting in an increase in both pressure and velocity. Since a large part of the energy of the water at discharge is kinetic, it is necessary to conserve this kinetic energy and transform it into pressure, if the pump is to be efficient.

As a matter of convenience in illustration, the water is represented as entering the impeller in Fig. 289 with a positive pressure. However, the pump is usually set above the level of the water from which it draws its supply, in which case the pressure at this point would be negative. Likewise, the axis of rotation need not be vertical as shown.

210. Classification.—Centrifugal pumps are broadly divided into two classes:

1. Turbine pumps.
2. Volute pumps.

While there are still other types, these two are the most fundamental. Also, as will be seen, these may, in turn, be subdivided in other ways.

The *turbine pump* is one in which the impeller is surrounded by a diffuser containing stationary guide vanes, as shown in Fig. 290. These provide gradually enlarging passages whose function it is to reduce the velocity of the water leaving the impeller and thus efficiently transform velocity head into pressure head. The casing surrounding the diffuser may be either circular and concentric with the impeller or spiral like the case of some reaction turbines.

The *volute pump*, shown in Fig. 291, is one that has no diffusion vanes, but, instead, the casing is of a spiral type so proportioned as to produce an equal velocity of flow all around the circumfer-

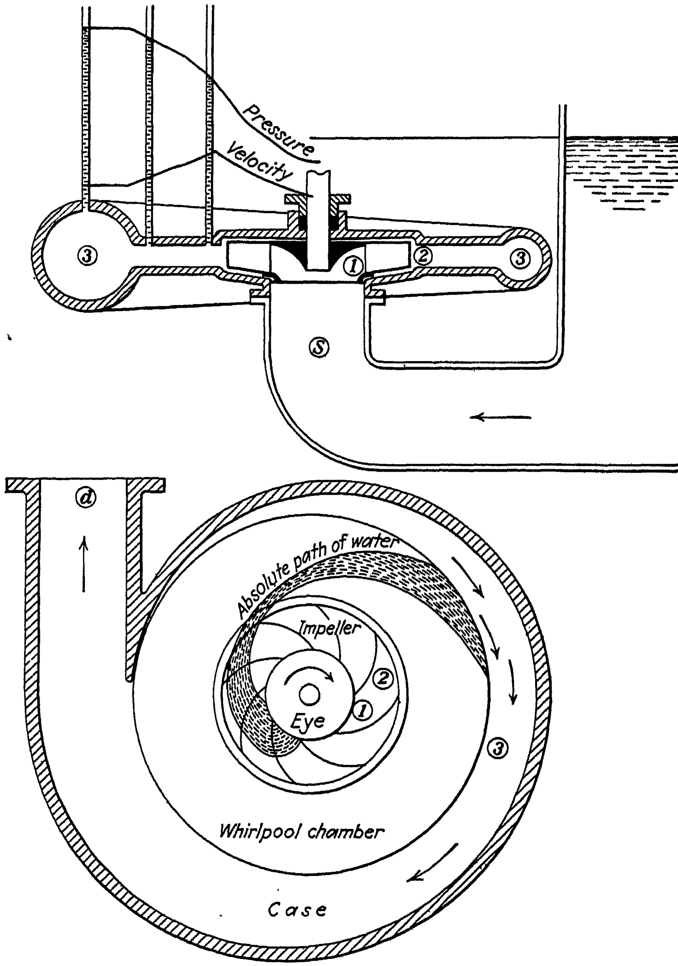


FIG. 289.

ence and also to reduce gradually the velocity of the water as it flows from the impeller to the discharge pipe. Thus the energy transformation is accomplished in a slightly different way. This spiral is often called a *volute*, whence the name of the pump.

Occasionally, pumps have been built with a whirlpool chamber, as shown in Fig. 289. This produces a free spiral vortex, the nature of which has been shown in Art. 58. This is an efficient means of converting velocity into pressure but is seldom employed

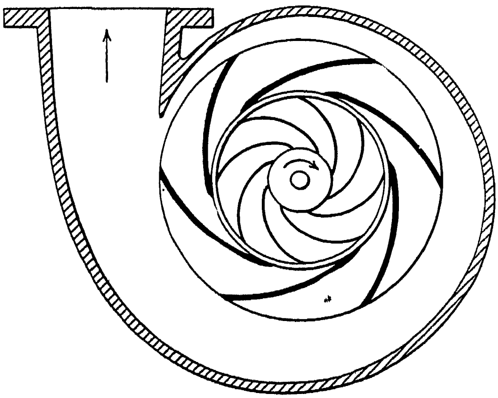


FIG. 290.—Turbine pump.

because it adds to the size of the pump and thereby increases its cost.

A very special type of centrifugal pump is the deep-well pump such as shown in Figs. 292 and 293. It is the same in theory as

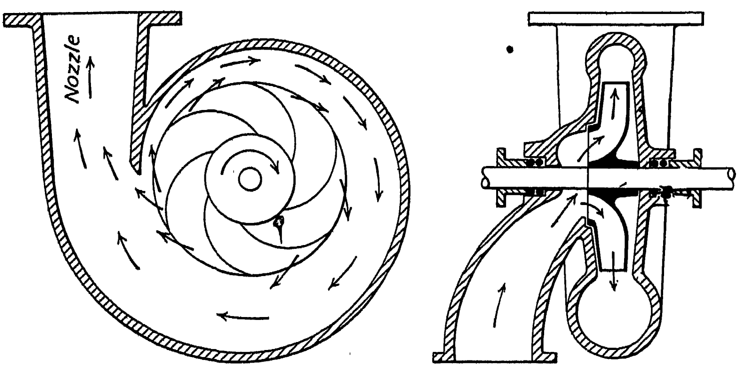


FIG. 291.—Volute pump.

any other centrifugal pump, but it differs considerably in mechanical construction. Since it must be installed in a well casing of limited size which sometimes extends 500 ft. or more below the surface, the entire pump assembly must be of small diameter. Because of the small diameter of the impellers it is usually neces-

sary to employ a number of them in series, as shown in Fig. 293, in order to lift the water to the surface. Also, the case surrounding the impeller must not be extended outward any more than is essential, and therefore the diffuser passages which contain the guide vanes are brought in toward the center, as shown in Fig. 293,

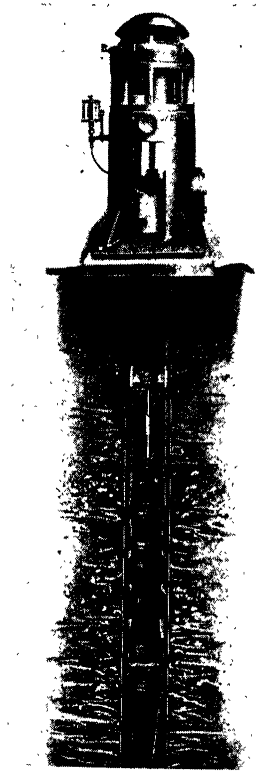


Fig. 292.—Deep-well pump. (*Courtesy of Byron Jackson Co.*)

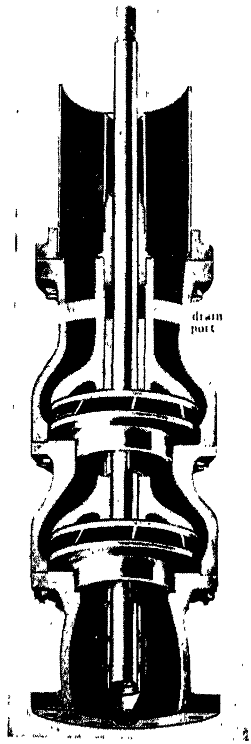


Fig. 293.—Two-stage deep-well pump. (*Courtesy of Byron Jackson Co.*)

and lead the water directly into the suction of the next impeller. Since these cases are usually concentric and are not of the volute type, it is customary to employ diffusion vanes, and hence deep-well pumps are generally the turbine type of centrifugal pump.

Another special form of pump is the propeller pump, in which the flow is axial instead of radial. It is very similar to the propeller turbine, such as shown in Fig. 257. Although centrifugal

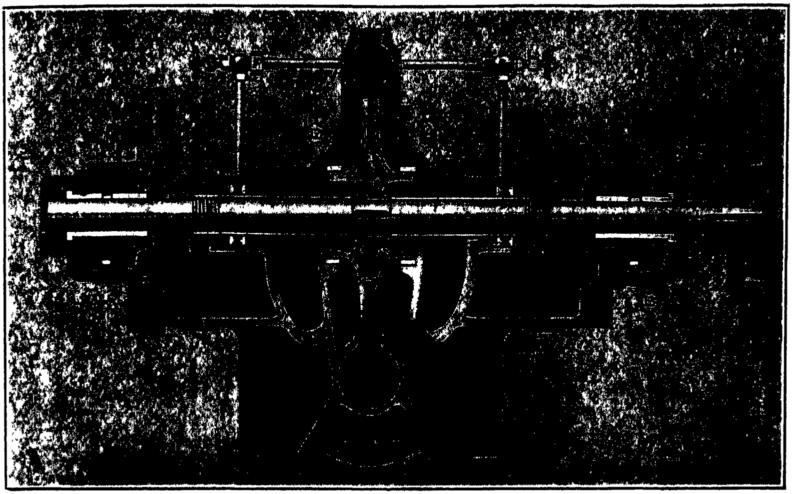


FIG. 294.—Double-suction volute pump. (*Courtesy of the Allis-Chalmers Mfg. Co.*)

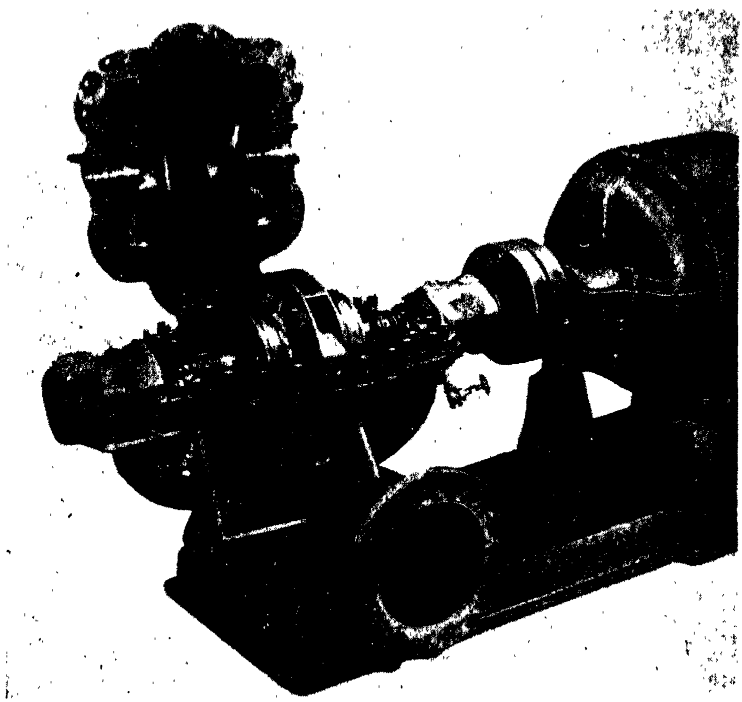


FIG. 295.—Double-suction pump. (*Courtesy of Byron Jackson Co.*)

action plays little or no part in its performance, it is included in this chapter because the general theory applies to it equally well. Different designs of propellers are shown in Figs. 299, 300, and 301.

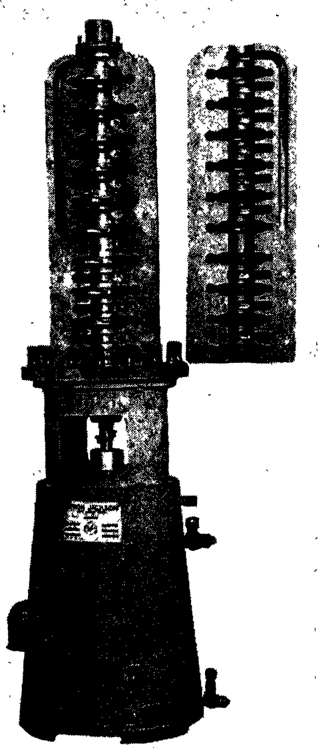


FIG. 296.—Multistage pump. (*Courtesy of Byron Jackson Co.*)

211. Description of the Centrifugal Pump.—The centrifugal pump is similar to the reaction turbine both in its construction and in its theory. However, one is not the reverse of the other, and their differences are as striking as their similarities.

The rotating part of the pump which is instrumental in delivering the water is called the *impeller*. Impellers may receive water on one side only, as in Fig. 291, in which case the pump is known as a single-suction pump, or, as in Figs. 294 and 295, from both sides, in which case it is known as a double-suction pump. A double-suction impeller is the same in effect as two single-suction impellers placed back to back. This construction has the effect of increasing the capacity without increasing the diameter of the impeller.

For high heads it becomes desirable to place impellers in series, as shown in Figs. 296 and 297, in which case the pump is called a multistage pump. Multistage pumps may be either of the turbine or of the volute type. Obviously, the addition of diffusion vanes results in a much more complex casing construction.

212. Conditions of Service.—Centrifugal pumps are used for lifting water to any height from a few feet only up to as much as several thousand feet. The greatest height to which water is lifted by a centrifugal pump is at the Grand Canyon of the Colorado where it is pumped 3,700 ft. vertically.

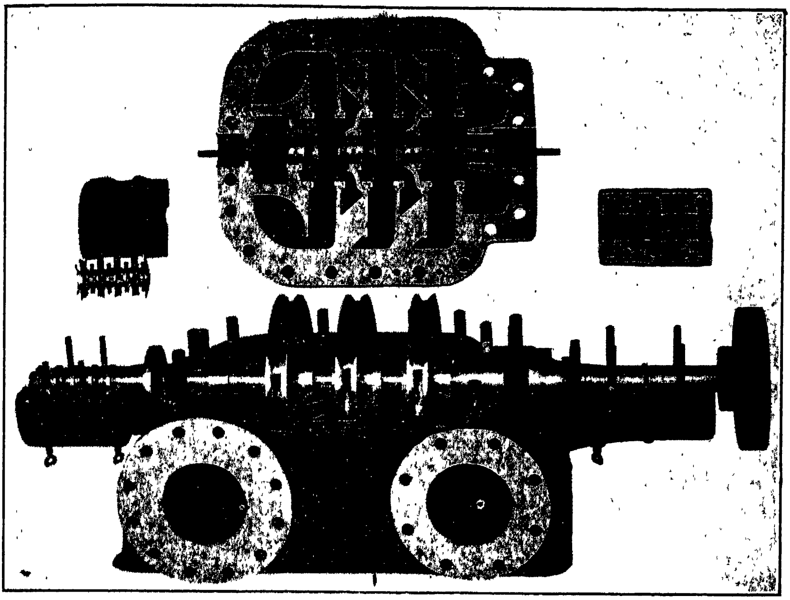


FIG. 297.—Three-stage centrifugal pump without diffusion vanes. (*Courtesy of Platt Iron Wks.*)

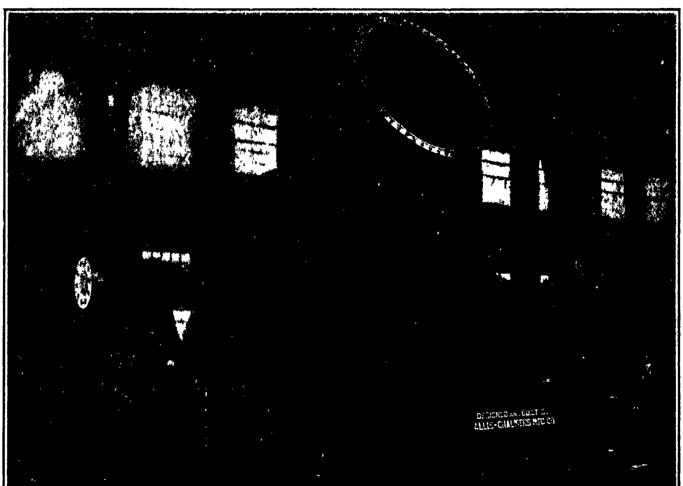


FIG. 298.—72-in. centrifugal pump for drainage at Memphis. $h = 15'$; $N = 100$. Capacity, 194,000,000 gal. per day. (*Courtesy of Allis-Chalmers Mfg. Co.*)

The capacities of centrifugal pumps range from very small quantities up to as much as 532 cu. ft. per sec., or 238,500 g.p.m.

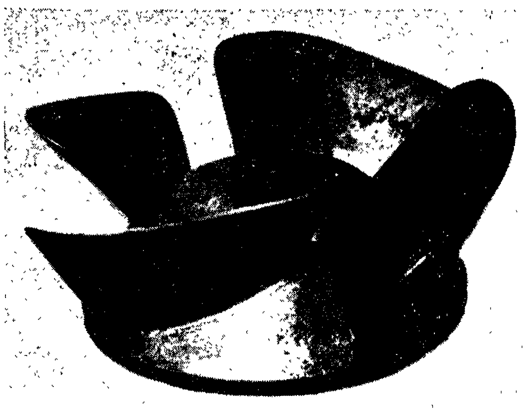


FIG. 299.—Moody spiral pump impeller. Specific speed = 9,000. (*Courtesy of Baldwin-Southwark Corp.*).

This latter is a German pump in which water is lifted to a height of 480 ft. by a two-stage pump running at 300 r.p.m. and requiring 34,200 hp. for its operation. It is the largest in the world so

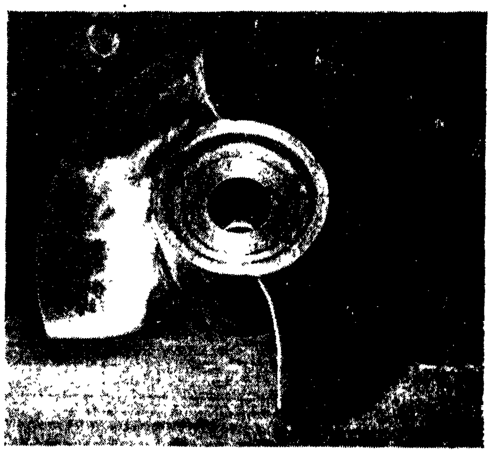


FIG. 300.—Propeller pump. (*Courtesy of Worthington Pump and Machinery Corp.*)

far but will be exceeded by those proposed for the Grand Coulee project which will lift water at the rate of 1,600 cu. ft. per sec. a height of 370 ft., thus requiring about 75,000 hp. per pump.

Rotative speeds may vary all the way from 30 to 3,000 r.p.m. in ordinary practice according to circumstances. The highest speed ever employed was 20,000 r.p.m. for a single-stage volute pump with an impeller 2.84 in. in diameter. The pump delivered 250 g.p.m. against a head of 700 ft. with an efficiency of 60.0 per cent. The highest peripheral speed used was with a single-stage pump with an impeller 3.15 in. in diameter. At 18,000 r.p.m. it delivered 189 g.p.m. against a head of 863 ft., and for a smaller discharge it developed a head of 995 ft.

Centrifugal pumps have been built with as many as 54 stages for very unusual conditions, but pumps of from 1 to 12 stages

FIG. 301.—Propeller pump. Specific speed = 10,000. (*Courtesy of De Laval Steam Turbine Co.*)

are much more common. It is customary to limit the head per stage to a value of about 100 to 200 ft., but this has been greatly exceeded in many cases. After careful experimental investigations made at the California Institute of Technology it was decided to use single-stage pumps for a lift of 444 ft. on the Colorado River Aqueduct. Each of these pumps delivers 200 cu. ft. per sec.

Centrifugal pumps have now been improved so that efficiencies of better than 90 per cent have been attained in some cases. The model pump for the Colorado River Aqueduct, whose performance is shown in Fig. 302, was found to have a maximum efficiency of 91.7 per cent at a head of 310 ft., which is the lift in the plant for which the full-size pumps are to be employed. The model pump built by Worthington Pump and Machinery Corporation for the 444-ft. lift plant on the aqueduct was found to have an efficiency of 90.9 per cent, while the model pump for the 155-ft.

lift built by Allis-Chalmers Manufacturing Company developed an efficiency of 92.6 per cent. These model pumps were all tested in a special laboratory at the California Institute of Technology, and the results are accurate to within 0.1 per cent.

Water turbines are rated according to the diameters of their runners, but the size of a centrifugal pump is usually designated by giving the diameter of the discharge pipe. The rated head and discharge for a centrifugal pump are the values for which

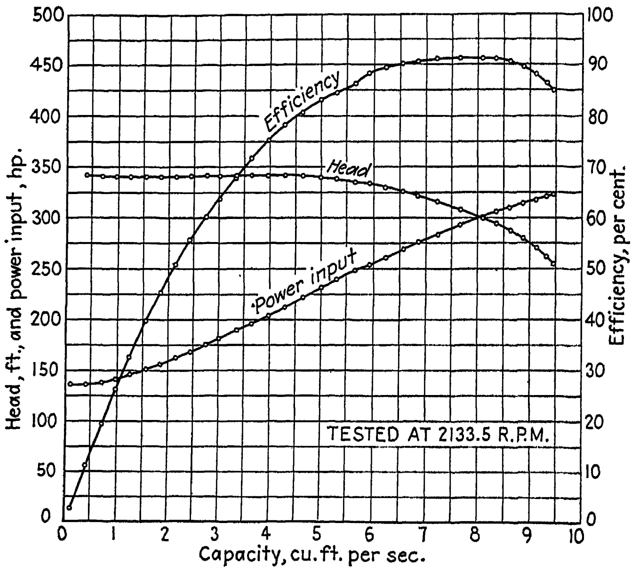


FIG. 302.—Byron Jackson centrifugal pump test at California Institute of Technology.

the efficiency is a maximum under a given speed. This value of the rate of discharge is often designated as the *normal discharge*. These values will be different for different speeds.

213. Head Developed.—The head developed by a centrifugal pump when no flow occurs is called the *shutoff head*, or the *head of impending delivery*. Its value may be found by applying the principles of Art. 57.

If water in a closed chamber is set in motion by a paddle wheel, as in Fig. 303, there will be an increase in pressure from the center to the circumference. If the water is assumed to rotate at the same speed as the impeller, the peripheral velocity of which is u_2, it may be seen from Eq. (49) that $p_2 - p_1 = wu_2^2/2g$, where

p_1 denotes the pressure at the center. If this water is in communication with a pressure chamber to which a piezometer tube is attached, as in Fig. 303, water will rise in the latter to such a height that

$$h_0 = \frac{u_2^2}{2g}. \qquad (261)$$

If the height of the tube were less than this, water would flow out, and the result would be a crude centrifugal pump.

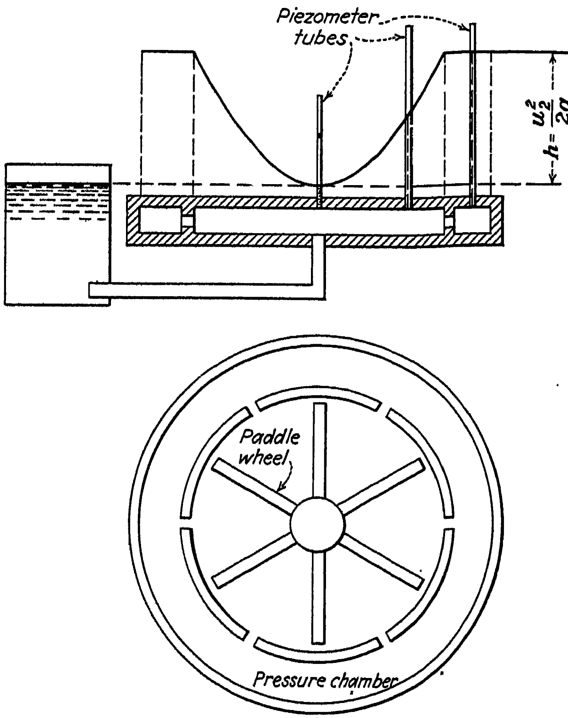


FIG. 303.—Crude centrifugal pump.

Actually certain influences are at work in the real pump which affect this relation slightly. Some of these factors tend to increase the head, and others to decrease it. The net effect is that for the usual type of centrifugal pump the head of impending delivery is

$$h_0 = 0.85 \text{ to } 1.10\,\frac{u_2^2}{2g}. \qquad (262)$$

But as soon as flow occurs, the foregoing relation is no longer true, as may be seen in Figs. 304 and 305. When water is being

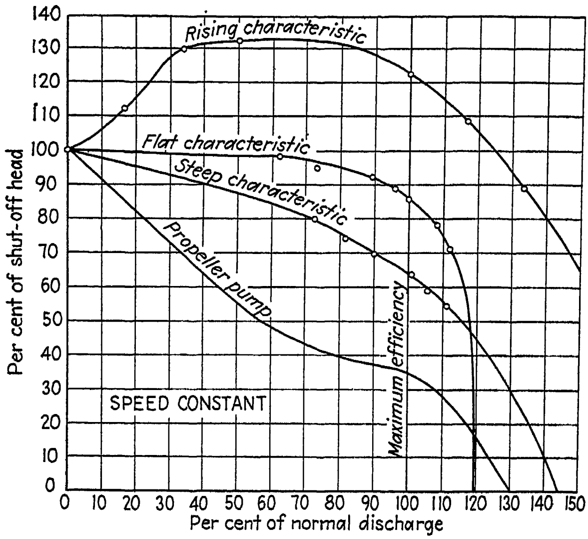


FIG. 304.—Head-discharge characteristics of different pumps.

delivered, the head may be either greater or less than the shutoff head, according to the design of the pump. A general equation

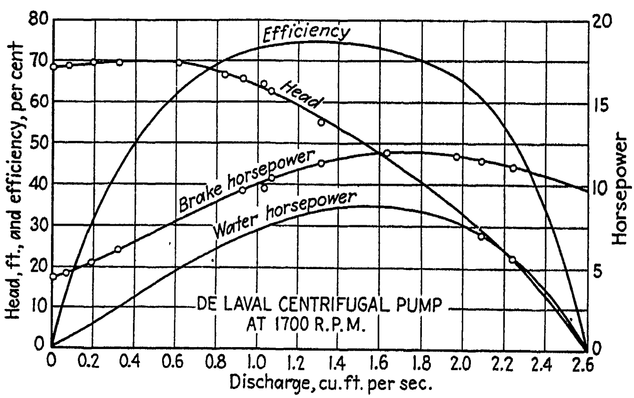


FIG. 305.—Characteristics of 6-in. pump at constant speed.

will now be derived relating head, impeller speed, and rate of discharge for all conditions of operation.

As the water in the suction pipe approaches the impeller, a rotary motion may be imparted to it before it ever reaches the

latter, due to the viscosity of intervening particles of water. Hence, equations will be written between points (2) and (s) in Fig. 289, the latter point being removed far enough from the impeller so that no rotational flow is imparted to the water.

In Fig. 306 is shown the hydraulic gradient in the case of zero flow, and, as in Eq. (261),

$$h_0 = \frac{u_2^2}{2g}.$$

When flow occurs, there will be a drop in pressure at (2), which is just within the impeller, due to the velocity head at that point

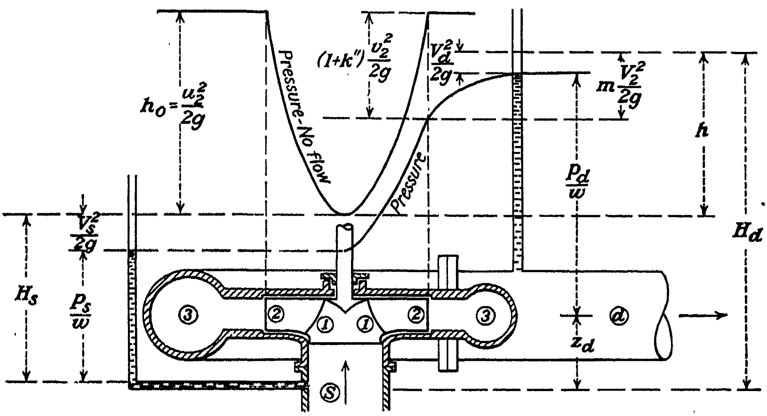


FIG. 306.

and also due to the loss of head within the impeller passages or from (s) to (2). If k'' is the coefficient of loss, the drop in pressure at the outlet of the impeller will be $(1 + k'')v_2^2/2g$.[1] But the pressure at (s) likewise decreases by an amount equal to $V_s^2/2g$. Also, as the water flows from (2) to (d) in Fig. 306, there is a reduction in velocity head from $V_2^2/2g$ to $V_d^2/2g$. This means a corresponding gain in pressure, though not without some

[1] As an illustration consider a hose with a nozzle on the end. When the nozzle is opened so that water may flow, the pressure at the base of the nozzle is decreased below the value obtained when it is closed, by an amount equal to the velocity head at that point and to the friction losses up to that point. If, next, the hose should be moved around, this pressure drop would not be affected in the least, for it is a function of the velocity of flow within the hose, which is the relative velocity, and does not depend upon the velocity of the water with respect to the earth.

loss. Thus $mV_2^2/2g$ is converted into pressure plus $V_d^2/2g$, the loss being $(1 - m)V_2^2/2g$.

From Eq. (265) the total head developed by the pump, including impeller and case, is (see Fig. 306)

$$h = H_d - H_s = \left(\frac{p_d}{w} + z_d + \frac{V_d^2}{2g}\right) - \left(\frac{p_s}{w} + z_s + \frac{V_s^2}{2g}\right)$$
$$= h_0 - \frac{(1 + k'')v_2^2}{2g} + \frac{mV_2^2}{2g}$$
$$= \frac{u_2^2}{2g} - (1 + k'')\frac{v_2^2}{2g} + \frac{mV_2^2}{2g}. \qquad (263)$$

In reality, there will be a further drop in pressure at (s) whe flow takes place, due to the loss of head in friction in the suctio pipe. However, this woul also have the effect of decrea ing the pressures at (2) and (d by the same amount. Henc the difference in pressure, with which we are here concerned, would be exactly the same.

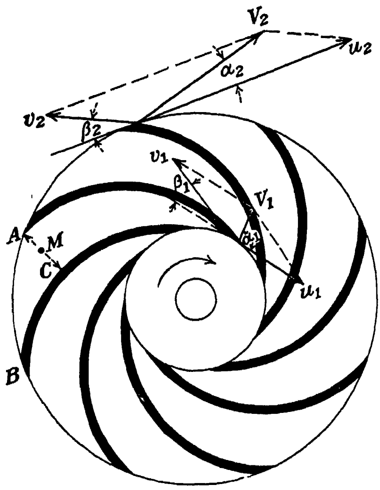


Fig. 307.

It must be noted that the quantity m is a variable. When the discharge from the impeller is such that the angle α_2 (see Fig. 307) agrees with the angle of the diffusion vanes of a turbine pump, or the velocity V_2 is the proper value for a volute pump, the maximum proportion of the velocity head will be saved. For larger or smaller discharges than this, additional losses will attend the conversion. For a turbine pump the maximum value of m is about 0.75, and for a volute pump it is somewhat less.

With a centrifugal pump the impeller areas are fixed and constant in value, and hence it is convenient to express the rate of discharge as

$$q = a_2 v_2. \qquad (264)$$

Now, referring to the vector diagrams shown in Figs. 307 and 308, it may be noted that as the rate of discharge varies the

values of V_2 and α_2 change. It may be seen that as q approaches zero, v_2 and α_2 approach zero, while V_2 approaches u_2. Hence for an infinitesimal discharge the value of V_2 may be regarded as equal to u_2, while the velocity of the water in the case surrounding the impeller is practically zero. Therefore, a particle of water leaving the impeller with a high velocity enters a body of ater at rest and loses all of its kinetic energy. Thus, as the ate of discharge approaches zero, the factor m approaches zero. Hence it may be seen that when there is zero discharge the value f h in Eq. (263) reduces to that given by Eq. (261).

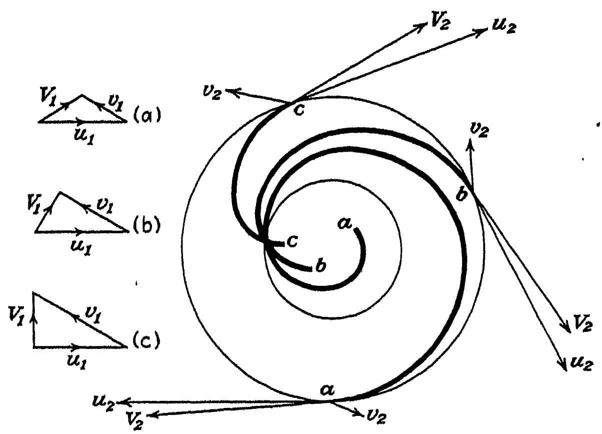


FIG. 308.—Streamlines for three different rates of discharge.

An inspection of Eq. (263) serves to explain the rising or falling characteristics of Fig. 304. If the increase of pressure due to the conversion of the velocity head of discharge is more than enough to offset the decrease due to the velocity and the losses within the impeller, a rising characteristic exists. If they are about equal, there is a flat characteristic; and if the quantity mV_2^2 is less than $(1 + k'')v_2^2$, there is a falling characteristic.

Because of the difficulty of efficiently transforming velocity head into pressure head it is desirable to keep V_2 as small as possible. It may be seen that, for a given value of v_2, the smaller the angle β_2 the less will be the magnitude of V_2. Therefore, in almost all centrifugal pumps the value of β_2 is from 20 to 30 deg., though occasionally this angle is as small as 10 or as large as

80 deg. It is rarely made larger than 90 deg. because of the inefficiency of such designs.[1]

214. Measurement of Head.—The head which a pump is required to work against may be computed by Eq. (170) or (172). The head which the pump can develop may be estimated by Eq. (263), but when it is desired to measure the head which the pump actually does develop it is done by taking certain readings on the discharge and suction sides of the pump. Thus in Fig. 309

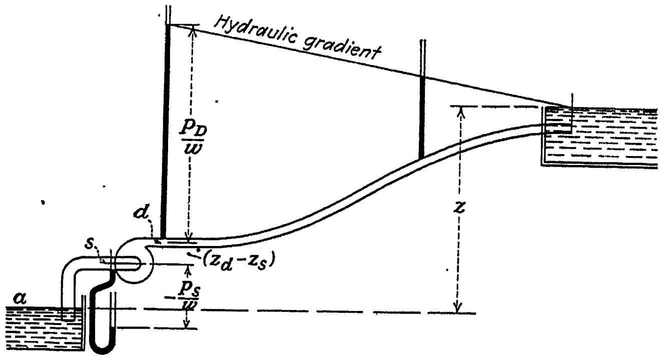


FIG. 309.—Head developed by pump.

the difference between the energy which the water has as it enters the pump at (s) and that with which it leaves at (d) is due solely to the pump. Hence,

$$h = H_d - H_s.$$

But

$$H_d = \frac{p_d}{w} + z_d + \frac{V_d^2}{2g},$$

and

$$H_s = \frac{p_s}{w} + z_s + \frac{V_s^2}{2g}.$$

Therefore, it follows that

$$h = \left(\frac{p_d}{w} - \frac{p_s}{w}\right) + (z_d - z_s) + \left(\frac{V_d^2}{2g} - \frac{V_s^2}{2g}\right). \qquad (265)$$

[1] In the turbine theory the angle β has been defined as the angle between v and u. This is satisfactory for that purpose, but with ordinary centrifugal pumps this angle is always greater than 90 deg. Hence it is much more convenient to define it here as the angle between v and $-u$, as may be seen in Fig. 307.

As a usual thing the water enters the pump under a pressure less than that of the atmosphere, in which case the value of p_s will be negative. If the suction and discharge pipes at the points where the gages are attached are of the same diameter, the velocity heads will cancel, in which event the value of h will be the difference in the levels of the surfaces of the two water columns shown in Fig. 309.

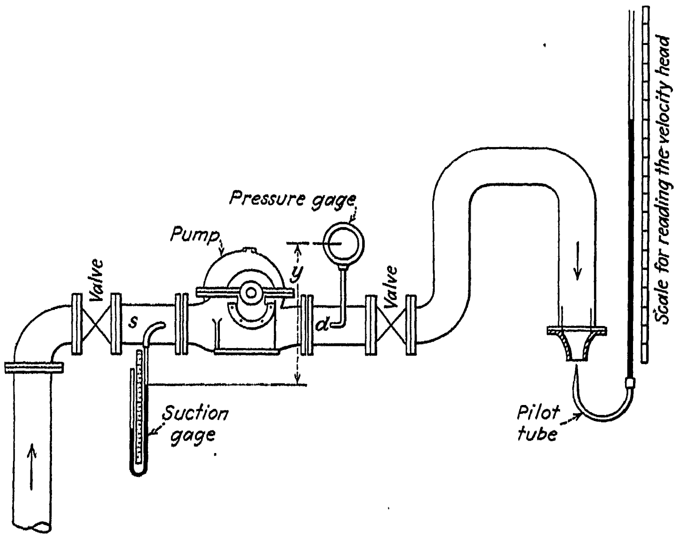


FIG. 310.—Measurement of head.

In testing the pump the gages might be connected as shown in Fig. 310. It is not really necessary to reduce the gage readings to the pressures that would be found at the center line of the pipe. If the gage readings are used direct in Eq. (265), and the value of y represents $(z_d - z_s)$, it may be shown that the result is the same.[1]

[1] The question is often raised as to why it is necessary to deduct $V_s^2/2g$ in determining the head, since the pump has imparted that velocity to the water. The first answer is that Eq. (265) is the result of a direct application of the principles of energy, but the explanation of the matter is that p_s, whose value is a function of V_s, has also been included. Suppose, for example, that the suction pipe were so large that the velocity in it was negligible. Then the measured value of p_s would give a higher pressure than when the suction pipe is smaller, and, disregarding losses, the values of p_s/w in the two cases would differ by $V_s^2/2g$. If the velocity head at s is to be omitted, the pressure reading should also be omitted. The total head

215. Head Imparted by Impeller.—The amount of energy delivered to the water by the impeller is greater than that actually delivered in the water, the difference being due to hydraulic

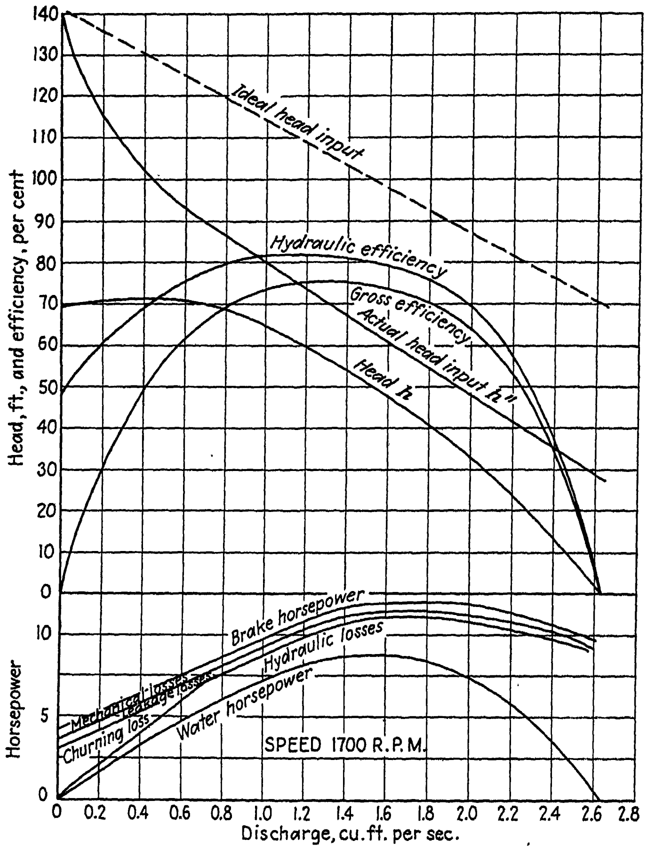


FIG. 311.—Analysis of centrifugal pump at constant speed.

friction losses within the pump. If the head actually developed by the pump is represented by h, the head imparted to the water by the impeller is

$$h'' = h + h',$$

---

might then be obtained by adding to the "discharge head" the value of z_s + *suction-pipe losses.* But the latter would have to be computed, and it may be shown that they are determined experimentally when p_s is measured and the velocity head $V_s^2/2g$ also employed.

or

$$h'' = \frac{h}{e_h}.$$

In an ideal pump without hydraulic losses of any kind these two quantities would be equal, but in any real pump they represent two entirely different things. For a given pump under different conditions of operation, h'' and h neither differ from each other by a constant amount, nor is one a constant proportion of the other. Hence the curves representing actual values of both h'' and h not only do not coincide, but they are not even of the same shape. This may be seen in Fig. 311 in which the curve *Actual head input* h'' has been determined with a reasonable degree of accuracy from test data by the author. It may be seen that for rates of discharge from 0 to 0.6 cu. ft. per sec. the value of h increases, while that of h'' decreases. In some other cases the difference is more marked than is here shown.

An expression for h'' may be derived by an application of the principles in Arts. 164 and 166. Since $h'' = u_2 V_2 \cos \alpha_2/g$, and $V_2 \cos \alpha_2 = u_2 - v_2 \cos \beta_2$,

$$h'' = \frac{u_2(u_2 - v_2 \cos \beta_2)}{g}. \tag{266}$$

This could also be obtained from Eq. (263) by eliminating the hydraulic losses. This would require values of $k'' = 0$ and $m = 1.0$. The next step would be to solve the vector triangle for V_2 in terms of u_2, v_2, and β_2. The result would agree with Eq. (266).

It may be seen that for a constant value of impeller speed the value of h'' given by Eq. (266) will increase with q(and v_2) for values of β_2 greater than 90 deg.; will be independent of q for $\beta_2 = 90$ deg.; and will decrease for values of β_2 less than 90 deg. It is sometimes argued from this that rising or falling characteristics are obtained by a suitable choice of β_2, but that is due to confusing h'' and h. The value of β_2 does have an effect upon this, but it alone does not determine the matter. The author has found a decidedly rising characteristic in a pump that he has tested with a vane angle of 26 deg. And tests of other pumps with a vane angle of 90 deg. have shown steep falling characteristics. The real explanation may be seen only in Eq. (263).

The hydraulic efficiency is the ratio h/h''. For the same reasons as are given in Art. 192, it is difficult to calculate true values of h'', and thus the *true* hydraulic efficiency can be determined only by test. The latter gives directly the value of the total efficiency only, and it is necessary to allow for other losses or determine them by special methods in order to get the actual hydraulic efficiency. Applying Eq. (266), and using the actual impeller dimensions, the computed values of h'' may be found to lie on a straight line, such as is labeled *Ideal head input* in Fig. 311. Thus the ratio between the actual h and the h'' computed in the ordinary manner is much less than the hydraulic efficiency in all cases. It is very often less than the gross effi-

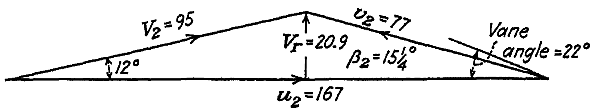


FIG. 312.—True velocity diagram as determined by direct measurements.

ciency, thus proving that it is not a true value. But it is still useful for some purposes of design and is called *manometric coefficient*. The value of this ratio is usually between 0.55 and 0.65.

The author stated in 1915 that the true value of the angle β_2 was less than the vane angle and that the two differed by about 5 to 10 deg. Such a correction would enable the computed value for h'' to agree with the experimental value.[1] This statement has recently been verified in the pump-testing laboratory at the California Institute of Technology.[2]

Thus Fig. 312 shows the actual velocity triangle at exit drawn from velocities and angles as directly measured in the laboratory from which the value of h'' was computed to be 486.5 ft. The actual head developed by the pump was 447.5 ft., which gives a hydraulic efficiency of 92 per cent. Since the over-all efficiency was found to be 88.5 per cent, this value for the hydraulic efficiency must be nearly correct. However, the vane angle was 22 deg. and was thus 6.75 deg. more than the actual direction of the water. Thus the theory is correct. The difficulty is in determining the proper values to use in the equations, as they

[1] "Centrifugal Pumps," p. 81.

[2] BINDER, R. C. and R. T. KNAPP, "Experimental Determinations of the Flow Characteristics in the Volutes of Centrifugal Pumps," *Trans. A.S.M.E.*, HYD 58, No. 4, p. 649, November, 1936.

do not coincide with those determined by measurements of the impeller.

216. Centrifugal-pump Factors.—Just as in the case of turbines, it is found that to obtain the best efficiency with a given centrifugal pump there must be a certain relation between head, speed, and discharge. Also, the equations show that these three quantities are mutually interrelated. Hence it may be seen from Eq. (263) that for a velocity diagram of the same shape to be formed it is necessary that u_2, v_2, and V_2 vary as $\sqrt{h}$. Hence should be written[1]

$$u_2 = \phi\sqrt{2gh}. \tag{267}$$

$$v_2 = c\sqrt{2gh}. \tag{268}$$

It is found that a certain value of ϕ is required to obtain the maximum efficiency, just as with the turbine. And a definite value of c is associated with every value of ϕ, as may be seen from Eq. (263). For ordinary types of pumps the following values of these factors are found:

For shutoff, $\phi = 0.95$ to 1.09.
For normal discharge, $\phi_e = 0.90$ to 1.30.
For normal discharge, $c_e = 0.10$ to 0.30.

The value of ϕ_e will depend upon the design of the pump. Thus the smaller the angle β_2 and the fewer the number of impeller vanes the larger the value of ϕ_e.

Just as in Art. 200, it may be shown that

$$N = \frac{1{,}840\phi\sqrt{h}}{D}. \tag{269}$$

217. Specific Speed.—The specific-speed factor for turbines involves the developed horsepower, since that is the quantity of particular interest. But with centrifugal pumps the primary interest is in their capacity, and it will be more useful if a similar expression, giving N_s in terms of discharge, is derived. Since power and discharge are really proportional to each other, it may be seen that the specific speed for the pump is merely being

[1] For the pump, ϕ has the same meaning as in the inward-flow reaction turbine, since it gives the peripheral speed in both cases. But c has a different meaning, since it is more convenient to deal with v_2 than with V_2.

expressed in terms of different units. Thus one may be derived from the other as follows:

$$N_s \text{ (turbines)} = \frac{N\sqrt{\text{b.hp.}}}{h^{5/4}} = \frac{N\sqrt{ewqh/550}}{h^{5/4}} = \sqrt{\frac{e}{8.81}} \times \frac{N\sqrt{q}}{h^{3/4}}.$$

Since the second portion of the last term involves the factors that are of primary interest with the pump, the specific speed for the centrifugal or propeller pump may be defined by it alone. While there might be a decided advantage in employing a quantity such as cubic feet per second, it is customary to rate pumps in gallons per minute, so that we shall write:

$$N_s \text{ (turbines)} = \frac{\sqrt{e}}{63} \frac{N\sqrt{\text{g.p.m.}}}{h^{3/4}},$$

which gives the numerical relationship between the turbine specific speed and the pump specific speed. Obviously, the specific speed for the centrifugal or propeller pump is

$$N_s = 21.2 \frac{N\sqrt{q}^*}{h^{3/4}} = \frac{N\sqrt{\text{g.p.m.}^*}}{h^{3/4}\dagger}. \tag{270}$$

For a multistage pump it is necessary to divide the total head by the number of stages to obtain the proper value of h for use in this equation, as well as practically all others in this chapter.

The physical interpretation of N_s for the pump is that it is the rotative speed at which a pump will run if it is of such a size as to deliver 1 g.p.m. at a head of 1 ft. per stage. But the real utility of it is that it is an index of the type of pump, as it is in the case of the turbine. Its use enables us to determine what combinations of head per stage, speed, and capacity are possible or desirable. If it is desired to employ a certain type of pump with a definite value of N_s, then the combination of these factors that is required may then be found.

The significance of specific speed as an index of type of pump is shown in Fig. 313, where a series of impellers is shown for the

* For a double-suction pump the rate of discharge used in this equation should be one-half the total pump capacity. The capacity as well as the head should be the values at the point of maximum efficiency for the value of N employed.

† Note that $h^{3/4} = h \div h^{1/4} = h \div \sqrt{\sqrt{h}}$. Some values of $h^{3/4}$ will be found on p. 448.

same head and capacity. It is seen that there is a great variation in the diameter at discharge and also a lesser variation in the diameter at entrance. The effect of the decreasing diameter at entrance is to require a higher entrance velocity and thus necessitate a lower suction lift if cavitation is to be avoided.

A value of specific speed of less than 500 is not practicable, and in fact a value even that small is undesirable. It is seen in Fig. 313 that as the specific speed increases, the pump impeller

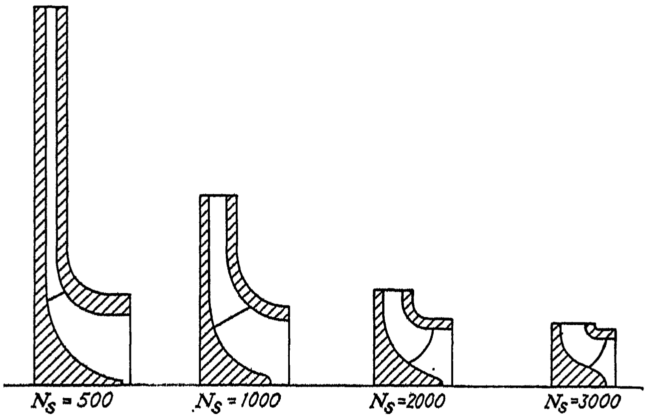


FIG. 313.—Profiles for same head and capacity but different specific speeds.

changes from the radial-flow to the mixed-flow type. At a sufficiently high value of N_s it would be necessary to employ the propeller pump.

218. Operation at Different Speeds.—In this chapter have been shown the characteristics of centrifugal pumps operating under variable heads at constant speeds. It is now desired to know how the pump is affected by a change in speed. This is shown by Eqs. (267) and (268). To obtain similar conditions of operation it is necessary that the values of ϕ and c be maintained constant. If they are, it may be seen that both the speed and the discharge of the pump will vary as the square root of the head. But if ϕ and c are not constant, then there is no simple index to the variation of the quantities. The only resort, then, is to a second-degree equation of the form shown in Art. 213. Hence if the head is varied due to a change in speed, it must be understood that the rate of discharge varies also if the following simple ratios are to apply.

From Eq. (267) it may be seen that

$$h = \frac{1}{\phi^2}\frac{u_2^2}{2g}, \tag{271}$$

which shows that if ϕ remains constant, the head developed varies as the square of the pump speed. From Eq. (268), after substituting the value of h given by Eq. (271), may be obtained

$$v_2 = \frac{c}{\phi}u_2. \tag{272}$$

Hence it follows that if ϕ remains constant, c will also remain constant, and the rate of discharge must vary directly as the speed. Since power is a function of the product of h and q, it may be seen that it will vary as the cube of the speed. The hydraulic efficiency will be constant as long as ϕ is constant, since all the hydraulic losses vary in the same way as the water horsepower. But the mechanical friction losses decrease slightly in percentage value as the speed increases, since they do not increase so rapidly. Hence the efficiency will really increase with the speed. But since the mechanical friction losses are such a small fraction of the total power involved, the change in the over-all efficiency is really very small.

But if the speed is increased, it means that the velocity of the fluid entering the "eye," or inlet, of the impeller will increase in proportion and thus that the intake pressure will decrease. Hence if the pressure is reduced sufficiently, cavitation will be induced. This will decrease the efficiency of the pump and also the head and capacity and thus alter the relations just stated. If the speed is further increased, eventually the cavitation will be found to fix a definite limit to the capacity, which cannot be exceeded. The only way to eliminate cavitation is to reduce the suction lift. It may even be necessary in some cases to supply the liquid under a positive pressure.

As long as cavitation is not encountered, the foregoing relations for a given pump may be summarized as

q varies as N;
h varies as N^2;
W.hp. varies as N^3;

and the efficiency increases very slightly with increasing N.

Sometimes a pump is of excess capacity and head for the speed that must necessarily be employed. An expedient often adopted is to turn down the impeller a small amount. The exact effect of this cannot be expressed in any simple way. The discharge will vary as a power of D between 1 and 2, while the head will vary approximately as D^2. The efficiency may be either reduced or increased depending upon the original relation between the impeller and the volute case in which it is mounted.

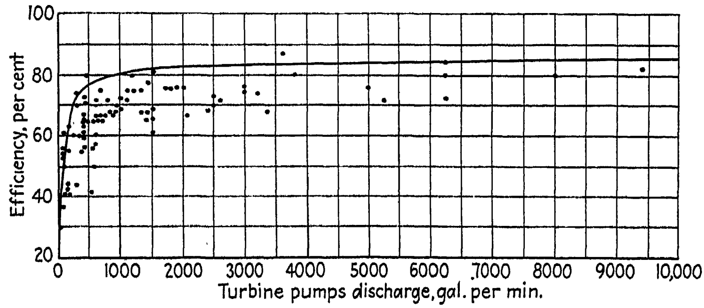


FIG. 314.—Optimum efficiency as a function of capacity.

For a series of homologous impellers of different sizes but of exactly the same design, if they are to run at the same r.p.m.,

$$h \text{ varies as } D^2;$$
$$q \text{ varies as } D^3;$$
$$\text{W.hp. varies as } D^5;$$

while the efficiency will increase slightly as the size increases. Thus the brake horsepower will not increase quite so much as the preceding expression indicates.

219. Factors Affecting Efficiency.—A most important factor in determining the efficiency of a centrifugal pump is its capacity, as may be seen in Figs. 314 and 315. While for any one capacity there is seen to be a decided range of values depending upon the skill of the designer, there is nevertheless a limit to the maximum values for each capacity, which increases very rapidly and then becomes substantially constant after a certain rate of discharge is reached.

These two diagrams, which are based upon the study of a large number of pumps, also show most clearly that volute pumps give slightly better efficiencies than turbine pumps. This is contrary

to the general impression, but it is well borne out by facts. For example, the Herdecke turbine pump with a capacity of 238,500 g.p.m. has an efficiency of 84.8 per cent, while the Rocky River volute pump with a capacity of 128,600 g.p.m. has an efficiency

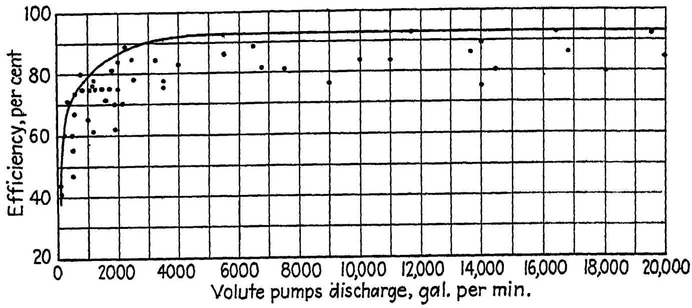


FIG. 315.—Optimum efficiency as a function of capacity.

of 91.9 per cent. The heads and other factors in the two cases were comparable.

The principal factor in reducing the efficiency of a small-size pump is the relatively large value of the leakage past the wearing

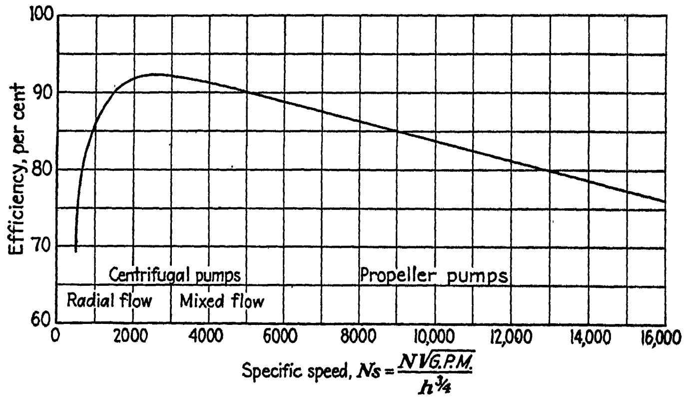


FIG. 316.—Optimum efficiency as a function of specific speed.

rings back into the suction side. Also, the disk friction of such a pump is a greater percentage of the total power expended.

It has been found that the effect of the head per stage is very slight, provided the design is suitable.

The effect of specific speed upon possible efficiency is important. An impeller with a low specific speed, as shown in Fig. 313,

will have a large disk friction and relatively large leakage losses. As the specific speed becomes larger, these losses diminish, and the efficiency increases, as shown in Fig. 316.[1] This curve represents what are believed to be optimum values for each specific speed. Of course, inferior designs, poor workmanship, and even good designs that must be adapted to unfavorable suction conditions will produce efficiencies lower than are to be found on this curve. But the diagram indicates the tendency of the efficiency to vary with the specific speed, when all other factors are the same. As larger specific speeds are encountered, the efficiency drops off owing to unfavorable hydraulic conditions. It is seen that for the very best results to be obtained the specific speed should be from about 2,000 to 3,500.

220. PROBLEMS

371. The curves of Fig. 311 are for a single-stage pump in which $D = 9.12$ in., $a_2 = 0.0706$ sq. ft., $\beta_2 = 27$ deg. At 1,700 r.p.m. when $q = 1.315$ cu. ft. per sec., $h = 55.7$ ft. If it is assumed that $m = 0.50$, find the value of k''. *Ans.* 5.69.

372. If it is assumed in Fig. 304 that the value of ϕ for shutoff is 1.0, what is the value of ϕ for the maximum lift of the pump with the rising characteristic? What is the value of ϕ for maximum efficiency in each case? *Ans.* 0.870, 0.902, 1.078, 1.250, 1.69.

373. The diameter of a pump impeller is 10 in. The speed is to be 1,200 r.p.m. If $\phi = 1.20$, what is the value of h? *Ans.* 29.5 ft.

374. Compute the value of the specific speed for (*a*) the pump shown in Fig. 298; (*b*) a two-stage pump which delivers 0.429 cu. ft. per sec. against a head of 225 ft. at 1,700 r.p.m. *Ans.* (*a*) 4,820; (*b*) 684.

375. What would be the capacity, head, and power of the pump whose performance is shown in Fig. 305, if it were run at a speed of 1,000 r.p.m.? What speed would be necessary to double the capacity of the pump? What speed would be required to double its lift?

Ans. 0.773 cu. ft. per sec., 19.2 ft., 2.36 hp., 3,400 r.p.m., 2,400 r.p.m.

376. If the speed of the pump of Fig. 305 were doubled, what would be the head for a discharge of 2.4 cu. ft. per sec.? What would be the efficiency for this rate of discharge at the higher speed? *Ans.* 236 ft., 0.74.

377. (*a*) It is desired to deliver 1,600 g.p.m. at a head of 900 ft. with a single-stage pump. What would be the minimum rotative speed that could be used? (*b*) It is desired to use a type of pump whose specific speed is 2,000 under a head of 16 ft. If the speed is to be 1,800 r.p.m., what will be the capacity? *Ans.* (*a*) 2,050 r.p.m. (*b*) 79 g.p.m.

378. If a speed of 600 r.p.m. is desired in Prob. 377 (*a*), how many stages must the pump have at least? *Ans.* 6.

[1] This curve has been supplied by A. Hollander, chief engineer, Byron Jackson Company.

379. Compute the specific speed of the pump whose performance is shown in Fig. 302. What would be the head, capacity, and power for this same pump if it were to run at 1,500 r.p.m.?

380. Assuming $\phi = 1$ at shutoff for the pump whose performance is shown in Fig. 302, what would be the diameter of the impeller? What would then be the value of ϕ_e?

381. If a pump of the same design as that for which Fig. 302 is constructed were to have linear dimensions six times as great and were to run at such a speed that it would develop the same head, what would be its r.p.m., capacity, and horsepower?

382. In the test of a centrifugal pump with water the rate of discharge was found to be 4,000 g.p.m., the pressure gage read 200 lb. per sq. in., the suction gage read 5 in. of mercury, and the difference in elevation between gages was 3 ft. The diameter of the suction pipe was 12 in., and that of the discharge pipe 8 in. If the brake horsepower was 600, what was the efficiency?

383. In a pump for which Fig. 312 applies, if the diameter of the impeller is 18 in., the width B 2 in., and the vanes take up 5 per cent of the space, what would be the rate of discharge?

384. It is desired to pump 100 cu. ft. of water per sec. against a head of 250 ft. What should be the r.p.m. of a pump to enable the maximum possible efficiency to be obtained?

385. What would be about the minimum and the maximum rotative speeds that could be employed for the case in Prob. 384, if efficiency were disregarded?

APPENDIX

Table XII.—Areas of Circles

Diameter		Area		Diameter		Area	
Inches	Feet	Square inches	Square feet	Inches	Feet	Square inches	Square feet
1/4	0.0021	0.0491	0.00034	30	2.500	706.9	4.90
1/2	0.0042	0.1963	0.00136	32	2.667	804.3	5.58
3/4	0.0062	0.4417	0.00306	34	2.830	907.9	6.30
1	0.083	0.7854	0.00545	36	3.000	1,018.0	7.07
1 1/4	0.104	1.227	0.00853	38	3.17	1,134.0	7.88
1 1/2	0.125	1.767	0.0123	40	3.44	1,257.0	8.72
1 3/4	0.146	2.405	0.0167	42	3.50	1,385.0	9.62
2	0.167	3.142	0.0218	44	3.67	1,521.0	10.57
2 1/2	0.208	4.909	0.0341	46	3.83	1,662.0	11.53
3	0.250	7.069	0.0492	48	4.00	1,810.0	12.56
3 1/2	0.292	9.621	0.0668	50	4.17	1,964.0	13.63
4	0.333	12.566	0.0872	52	4.33	2,124.0	14.75
4 1/2	0.375	15.909	0.1105	54	4.50	2,290.0	15.90
5	0.417	19.635	0.1362	56	4.67	2,463.0	17.10
6	0.500	28.27	0.196	58	4.83	2,642.0	18.35
7	0.583	38.48	0.267	60	5.00	2,827.0	19.62
8	0.667	50.26	0.349	62	5.17	3,019.0	20.92
9	0.750	63.62	0.442	64	5.33	3,217.0	22.3
10	0.833	78.54	0.545	66	5.50	3,421.0	23.8
12	1.000	113.1	0.785	68	5.67	3,632.0	25.2
14	1.167	153.9	1.068	70	5.83	3,848.0	26.7
16	1.333	201.1	1.395	72	6.00	4,072.0	28.3
18	1.500	254.5	1.765	76	6.33	4,536.0	31.4
20	1.667	314.2	2.18	80	6.67	5,027.0	34.9
22	1.833	380.1	2.64	90	7.50	6,362.0	44.2
24	2.000	452.4	3.14	100	8.33	7,854.0	54.5
26	2.164	530.9	3.68	110	9.17	9,503.0	66.0
28	2.332	615.8	4.27	120	10.0	11,310.0	78.5

TABLE XIII.—STANDARD WROUGHT-IRON PIPE SIZES

Diameter		Internal area		Diameter		Internal area	
Nominal, inches	Actual internal, inches	Square inches	Square feet	Nominal, inches	Actual internal, inches	Square inches	Square feet
$\frac{1}{8}$	0.27	0.0573	0.0004	3	3.067	7.388	0.0513
$\frac{1}{4}$	0.364	0.1041	0.0007	$3\frac{1}{2}$	3.548	9.887	0.0687
$\frac{3}{8}$	0.494	0.1917	0.0013	4	4.026	12.73	0.0884
$\frac{1}{2}$	0.623	0.3048	0.0021	5	5.045	19.99	0.1388
$\frac{3}{4}$	0.824	0.5333	0.0037	6	6.065	28.89	0.2006
1	1.048	0.8626	0.0060	7	7.023	38.74	0.2690
$1\frac{1}{4}$	1.380	1.496	0.0104	8	7.982	50.04	0.3474
$1\frac{1}{2}$	1.611	2.038	0.0141	9	8.937	62.73	0.4356
2	2.067	3.356	0.0233	10	10.019	78.84	0.5474
$2\frac{1}{2}$	2.468	4.784	0.0332	12	12.000	113.1	0.7854

TABLE XIV.—VALUES OF $m^{2/3}$

m	$m^{2/3}$	m	$m^{2/3}$	m	$m^{2/3}$	m	$m^{2/3}$
0.2	0.342	2.2	1.69	4.0	2.52	8.0	4.00
0.4	0.543	2.4	1.79	4.2	2.60	8.5	4.17
0.6	0.712	2.6	1.89	4.5	2.73	9.0	4.33
0.8	0.863	2.8	1.98	5.0	2.92	10.0	4.63
1.0	1.000	3.0	2.08	5.5	3.12	11.0	4.93
1.2	1.13	3.2	2.17	6.0	3.29	12.0	5.22
1.4	1.25	3.4	2.26	6.5	3.48	13.0	5.52
1.6	1.37	3.6	2.35	7.0	3.66	14.0	5.80
1.8	1.48	3.8	2.44	7.5	3.83	15.0	6.10
2.0	1.58						

TABLE XV.—VALUES OF $h^{3/4}$

h	$h^{3/4}$	h	$h^{3/4}$	h	$h^{3/4}$	h	$h^{3/4}$
10	5.62	25	11.18	70	24.20	140	40.6
11	6.03	30	12.82	80	26.77	150	42.8
12	6.45	35	14.38	90	29.33	170	47.1
13	6.85	40	15.90	100	31.6	200	53.2
14	7.24	45	17.38	110	33.9	230	59.0
16	8.00	50	18.80	120	36.2	260	64.8
18	8.73	60	21.25	130	38.5	300	72.0
20	9.45						

FUNDAMENTAL TRIGONOMETRY

In a right-angle triangle, such as Fig. 317,

$$\sin A = \frac{a}{c}. \qquad \sec A = \frac{c}{b}.$$

$$\cos A = \frac{b}{c}. \qquad \csc A = \frac{c}{a}.$$

$$\tan A = \frac{a}{b}. \qquad \cot A = \frac{b}{a}.$$

Any function of A is the same numerically as the cofunction of any combination of A with an odd multiple of 90 deg. Thus:

$$\sin A = \cos (90 \text{ deg.} \pm A) = \cos (270 \text{ deg.} \pm A).$$

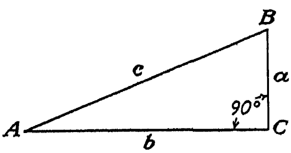


Fig. 317.

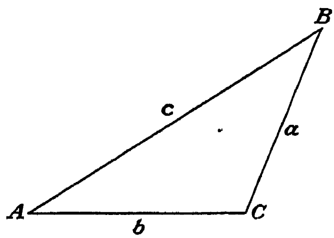


Fig. 318.

Any function of A is the same numerically as the function of any combination of A with an even multiple of 90 deg. Thus:

$$\sin A = \sin (180 \text{ deg.} \pm A).$$

The sign of the function depends in any case upon the quadrant in which the angle itself lies.

For the solution of an oblique triangle, such as that shown in Fig. 318,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$a^2 = (b - c)^2 + 4bc \sin^2 \frac{A}{2}.$$

$$a^2 = (b + c)^2 - 4bc \cos^2 \frac{A}{2}.$$

$$a^2 = (b \sin A)^2 + (c \cos A - b)^2.$$

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